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Perturbation Theory of the Fermi Surface in a Quantum Liquid. A General Quasiparticle Formalism and One-Dimensional Systems

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We develop a perturbation theory formalism for the theory of the Fermi surface in a Fermi liquid of particles interacting via a bounded short-range repulsive pair potential. The formalism is based on the renormalization group and provides a formal expansion of the large-distance Schwinger functions in terms of a family of running couplings consisting of one- and two-body quasiparticle potentials. The flow of the running couplings is described in terms of a beta function, which is studied to all orders of perturbation theory and shown to obey, in the nth order, $n!$ bounds. The flow equations are written in general dimension $d \geqslant 1$ for the spinless case (for simplicity). The picture that emerges is that on a large scale the system looks like a system of fermions interacting via a δ-like interaction potential (i.e., a potential approaching 0 everywhere except at the origin, where it diverges, although keeping the integral bounded); the theory is not asymptotically free in the usual sense and the freedom mechanism is thus more delicate than usual: the technical problem of dealing with unbounded effective potentials is solved by introducing a mathematically precise notion of *quasiparticles*, which turn out to be natural objects with finite interaction even when the physical potential diverges as a deltalike function. A remarkable kind of gauge symmetry is associated with the quasiparticles. To substantiate the analogy with the quasiparticle theory we discuss the mean field theory using our notion of quasiparticles: the resulting self-consistency relations are closely reminiscent of those of the BCS model. The formalism seems suited for a joint theory of normal states of Fermi liquids and of BCS states: the first are associated with the trivial fixed point of our flow or with nearby nontrivial fixed points (or invariant sets) and the second may naturally correspond to really nontrivial fixed points (which may nevertheless turn out to be accessible to analysis because the BCS state is a quasi free state, hence quite simple, unlike the nontrivial fixed points of field theory). The $d = 1$ case is deeply different from the $d > 1$ case, for our spinless fermions: we can treat it essentially completely

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for small coupling. The system is not asymptotically free and presents anomalous renormalization group flow with a vanishing beta function, and the discontinuity of the occupation number at the Fermi surface is smoothed by the interaction (remaining singular with a coupling-dependent singularity of power type with exponent identified with the anomalous dimension). Finally, we present a heuristic discussion of the theory for the flow of the running coupling constants in spinless $d > 1$ systems: their structure is simplified further and the relevant part of the running interaction is precisely the interaction between pairs of quasiparticles which we identify with the Cooper pairs of superconductivity. The formal perturbation theory seems to have a chance to work only if the interaction between the Cooper pairs is repulsive: and to second order we show that in the spin-0 case this happens if the physical potential is repulsive. Our results indicate the possibility of the existence of a normal Fermi surface only if the interaction is repulsive.

KEY WORDS: Perturbation theory; Fermi surface; quantum liquid

0. INTRODUCTION. RESULTS

We consider a system of fermions at zero temperature, with mass $m > 0$ and positive density, interacting via a smooth, short-range, rotation-invariant pair potential (described in an appropriate grand canonical ensemble): hence, hard-core interactions are not considered. The fermions will be usually supposed spinless: the simplification introduced by the absence of spin is, in the perturbation theory part of this paper, unessential. We also treat, as an example, a spin case in a special one-dimensional model.

We shall most of the time introduce a second simplification, namely, we shall suppose that there is a short-range cutoff eliminating scales shorter than some length scale p_0^{-1}: the best way to do this would be to imagine that our fermions are on a lattice. However, such a regularization would not be spherically symmetric and the resulting complications would hide the conceptual problems and our proposals for their solutions; hence, our regularization will be spherically symmetric in space time. We think that the above simplification, although unphysical, is unessential and that the ultraviolet problem that is eliminated in this way could in fact be solved and that it has nothing to do with the theory of the Fermi surface, which is an infrared problem. For this reason, in developing the general theory, we have not even attempted the analysis of the ultraviolet problem and we have fixed p_0^{-1} equal to the range of the interaction potential. We devote attention to the ultraviolet problem only in the $d = 1$ spinless case with repulsive interaction.

The basic question is to prove that, under convenient assumptions on

the pair potential, the interacting system has a well-defined normal Fermi surface. The first problem is to provide a definition of a normal Fermi surface: the guide is the theory of the free case (0 interaction). A Fermi surface with radius p_F (which in free systems is trivially related to the chemical potential or to the particle density; see Section 1) manifests itself in several ways. We consider the two-point Schwinger function in Euclidean space time[10]; if its argument $(\mathbf{x}, t) \equiv \xi$ is led to ∞, it behaves in d-space dimensions as

$$S_{p_0}(\mathbf{x}, t) = Z^{-1} \int \frac{dk_0\, d\mathbf{k}}{(2\pi)^{d+1}} \frac{e^{-i(k_0 t + \mathbf{kx})}}{-ik_0 + (\mathbf{k}^2 - p_F^2)/2m} \vartheta\left(\frac{k^2}{p_0^2}\right)$$
$$\xrightarrow[|\xi| p_F \gg 1]{} \frac{p_F^{d-1} \Omega_d}{(2\pi)^d \beta Z} \int d^{d-1}\boldsymbol{\omega}\, \frac{t - i\beta^{-1}\boldsymbol{\omega}\mathbf{x}}{t^2 + \beta^{-2}\mathbf{x}^2} e^{i\boldsymbol{\omega}\mathbf{x} p_F} \tag{0.1}$$

where $\vartheta(x)$ is a cutoff function taking out the ultraviolet part of the Schwinger function S, Ω_d is the surface of the d-dimensional unit sphere and $d^{d-1}\boldsymbol{\omega}$ is the normalized surface element on the sphere, $\beta = p_F/m$ is the velocity at the Fermi surface, $Z = 1$, and p_0^{-1} is a unit of length fixed arbitrarily. The parameter Z is introduced for later reference and will be called the *Fermi surface discontinuity* parameter (see Section 3 and Appendix A).

The reason for the latter name is that, if $p_0 = \infty$, the integral in (0.1) over k_0 can be performed by residues and one finds

$$S_\infty(\mathbf{x}, 0^-) = \frac{Z^{-1}}{(2\pi)^d} \int d^d\mathbf{k}\, \chi(\mathbf{k}^2 - p_F^2 < 0)\, e^{i\mathbf{kx}} \tag{0.2}$$

where $\chi(x < 0) = 1$ if $x < 0$ and vanishes otherwise, while the constant Z is identically 1; one would like to show that the presence of interaction simply replaces in (0.2) χ with $Z^{-1}\chi + \bar{\chi}$, with $\bar{\chi}$ regular, so that Z measures the jump at the Fermi surface; see ref. 9.

As one can certainly imagine, this is by no means the only possible definition of a Fermi surface: in fact, in Section 4 we present another way of defining it and at the end of Section 5, after (5.33), still more ways. All the definitions agree in the case of no interaction. In the presence of interaction they are formally equivalent. The proof of actual equivalence would require a lot of work: it is reminiscent of the problem of the equivalence of the ensembles in statistical mechanics. We shall eventually adopt the last of the possible definitions that we consider [see the analysis following (5.33) in Section 5] because it is the simplest to reach from scratch and, as we would like to believe, it is also the most fundamental, as it requires, to be stated, the introduction of the notion of *quasiparticles*, which is a basic

notion in low-temperature physics (but which so far has hardly received a mathematically useful definition; see, however, ref. 4).

The results of this paper can be loosely stated (and they acquire a precise meaning after the notion of quasiparticle is introduced) as follows.

1. We give a general perturbation-theory setup for the Fermi surface problem, with bounds on the beta function to all orders. This part of the work (Sections 6–12 and the nonheuristic part of Section 14) does not use in an essential way the spinless nature of our fermions and it can be trivially extended to spinning fermions.

2. There is a lack of asymptotic freedom if $d=1$. The beta function structure is inconsistent with a normal Fermi surface. We therefore introduce and discuss, for spinless systems, the notion of anomalous Fermi surface and build the new notions of effective potential and of beta function.

3. We introduce a certain technical property which we conjecture [see Section 15, before (15.48)] to hold for the exactly soluble model of Luttinger. Unfortunately, we have not been able to show that it is a consequence of the exact solution of the model. *Assuming* the conjecture *and* the analyticity of the beta function near the origin (see below), we deduce, from the known properties of the exact solution by Mattis and Lieb of Luttinger's model,[25,26] that the anomalous beta function is identically zero in the spinless case. And we show that in turn this implies that the general short-range interaction leads to an abnormal Fermi surface identical, qualitatively, to the one present in the Luttinger model (Section 15). In this case we outline a discussion of the ultraviolet problem mentioned at the beginning of this section under the additional assumption that the potential is repulsive in the strong sense of being positive definite.

4. The analyticity of the beta function in the $d=1$, spinless (anomalous or not) or spinning case should be technically a consequence of the estimates in ref. 19: in fact, it is clear from our analysis that the analyticity properties of the beta function should be the same for our models and for the 2-dimensional Gross–Neveu model. Gawedski and Kupiainen[19] discuss the running couplings flow and the analyticity properties without defining exactly the beta function in the same sense of refs. 16–18, which we use here: we think that this was only an expository choice and we plan to study formally this technical point in a separate paper.

5. For spinning fermions we have no results beyond the ones in item 1 above. We discuss briefly the $d=1$ spinning case (Section 15), pointing out some obvious problems, but providing no solutions. It is described by a map in finite dimension which contains resonances and therefore its

theory seems to lead quickly to well-known unsolved problems (small divisors, diffusion, etc.). This has something to do, we believe, with the fact that in the spinning $d=1$ case a deltalike interaction with nonvanishing integral may be nontrivial and therefore boundedness of the running form factors is *per se* not sufficient to solve the problem.

6. If $d>1$ the results are not sufficient to show the consistency of the theory even to second order (in the sense of Section 5), even though we control in some sense the beta functional to all orders (Section 14). We hope that consistency to second order can be checked with extra work. We write down the equation which should be discussed, describing some of its elementary properties and its connection with the adopted Fermi surface definition. We also present some heuristic analysis of it (in the spinless cases), based on the idea that the flow generated by the beta function is governed by a function describing what we call the interaction between Cooper pairs. The analysis leads to a flow with *no anomalous dimension*, if the interaction is repulsive: thus, such spinless systems with repulsive interaction would have a normal Fermi surface.

We now sketch the logical structure behind the technical work.

In the interacting theory, keeping the range p_0^{-1} of the interaction fixed, one has three independent parameters, namely the potential λ_0 (which is in fact a function), the mass m, and the p_F (or $\beta \equiv p_F/m$), which is the radius of the Fermi surface. The discontinuity has to be determined (when existing). Given the interaction potential λ_0, and some *bare* values m_0 and p_F^0 of the mass and of the chemical potential, by definition we shall say that our system is normal and has particles of mass m, Fermi surface at p_F, and discontinuity Z if the pair Schwinger function S_{p_0} (cut off at p_0) has the asymptotic behavior (0.1) or if some formally equivalent property holds.

Formula (0.1), in the approximation expressed by the rhs, has remarkable scaling properties suggesting the use of *renormalization group* methods. In fact, the rhs of (0.1), and even the first nonapproximate expression generating it, will be shown to determine a representation of the Fermi field describing in Euclidean space time the free ground state with parameters p_F, β of the type

$$\psi^{\pm}_{\xi, p_F, \beta} = \sum_{n=-\infty}^{0} \psi^{n,\pm}_{\xi, p_F, \beta} \tag{0.3}$$

where ψ^n is a field *on scale* $2^{-n}p_0^{-1}$ in the sense of renormalization group (called the *component* on scale n or, in case of ambiguity, on scale $2^{-n}p_0^{-1}$), and in fact it has a scale covariance property:

$$\psi^{n}_{\xi, p_F, \beta} \approx 2^{dn/2}\psi^{0}_{2^n\xi, 2^{-n}p_F, \beta}, \qquad n \to -\infty \tag{0.4}$$

We mention this property here because it is natural to study the scaling properties of the fields $\psi^{n,\pm}$ and to try to see what can be done with them: however, the discussion below and, in more detail, Sections 4 and 10 make it clear that, in fact, the scaling property (0.4), although correct, is not very useful and it is quite misleading.

The decomposition (0.3) transforms the problem of computing the Schwinger functions into that of the theory of functional integrals of the exponential of an action $V(\psi^{\pm})$ with respect to the Euclidean Fermi fields ψ^n (see Sections 1–4 and 6).

The natural approach (*renormalization group approach*) would be to integrate successively the field components defining recursively the *effective potentials* $V^{(h)}(\psi^{(\leqslant h)})$, where $\psi^{(\leqslant h)}$ is defined by the sum in (0.3) stopped at $h \leqslant 0$. One has to identify in $V^{(h)}$ a *relevant part* depending on few parameters and an *irrelevant* part, the remainder. In our case the naive candidates would be a quartic operator in the fields which has the same form as the pair potential operator plus a quadratic part.

In this way the relevant part $V_L^{(h)}$ of $V^{(h)}$ takes the form

$$\int_\Lambda \lambda_h(\xi-\eta)\,\psi^{(\leqslant h)+}_{\xi,t}\psi^{(\leqslant h)-}_{\xi,t}\psi^{(\leqslant h)+}_{\eta,t}\psi^{(\leqslant h)-}_{\eta,t}\,d\xi\,d\eta\,dt$$

$$+\int_\Lambda 2^h v_h \psi^{(\leqslant h)+}_{\xi}\psi^{(\leqslant h)-}_{\xi}\,d\xi+\int_\Lambda \frac{\alpha_h}{2m}\,(-\partial_\xi^2-p_F^2)\,\psi^{(\leqslant h)+}_{\xi}\psi^{(\leqslant h)-}_{\xi}\,d\xi$$

$$+\int_\Lambda \zeta_h \psi^{(\leqslant h)+}_{\xi}\partial_t\psi^{(\leqslant h)-}_{\xi}\,d\xi \tag{0.5}$$

where the coefficient of the $\psi^+\psi^-$ term has been defined as $2^h v_h$ rather than v_h for later convenience. It is, however, nontrivial to identify the relevant part contribution inside $V^{(\leqslant h)}$. The coefficients of the relevant part are called *running couplings* or *form factors* (note that one of them is a function).

The identification of the relevant part should be such that there is the possibility of expressing both the relevant and irrelevant parts of the effective potential on scale h as formal power series in the higher scales, $h' > h$, running couplings: here the h's are negative, which is very convenient and not as confusing as it looks at first sight. Furthermore, one demands:

1. If the running couplings are supposed bounded uniformly in h, then the sum of the absolute values of the nth-order coefficients of the above formal expansions should be bounded by $C^n n!$, where $n!$ is an estimate of the number of Feynman diagrams (with n vertices) in the classical perturbation theory; see refs. 1–9 and the brief introduction in Sections 1–4. This is usually interpreted as

saying that there are no divergences in the theory, *other than* the ones associated with the running couplings themselves, when one tries to expand the effective potential (or the Schwinger functions which are trivially related to it) in powers of the 0-scale running couplings.

2. If the expansions expressing the running couplings on scale h in terms of the ones on higher scales are truncated to any pre-fixed order p, then they generate a sequence of running couplings which is indeed uniformly bounded in h, at least for some suitably chosen nontrivial initial values.

The first property will be called the *existence of the beta function* of the theory and the second *consistency to order p* of perturbation theory.

If, furthermore, the Schwinger functions, computed to the same order in the running couplings are such that, to the considered order p, they obey the asymptotic relation defined in (0.2) or one of the formally equivalent relations examined in Sections 4 and 5, one says that the Fermi surface *exists to order p and is normal.*

Usually this is obtained by showing that the effective potentials tend to zero in the considered order (*asymptotic freedom*). But this is by no means necessary: in fact, only some of the running couplings enter into (0.2). The theory of the Fermi surface seems to provide an example of the above nonnecessity.

With the above program in mind, one starts computing the flow of the effective potential coupling constants to second order: one is deceived in discovering that no matter how one proceeds (at least, no matter how *we* proceeded), one finds that the effective potential has the annoying feature of trying to diverge.

The latter property makes it impossible to proceed with the usual techniques[16–18] which are based on uniform estimates. The difficulty is a major one: it is basically linked to the fact that the problem has an intrinsic scale of length built in, p_F^{-1}; the latter introduces oscillations in the integrands expressing the effective potentials, which produce cancellations compensating some of the divergences caused by the size of the effective potentials, the others being eliminated ultimately by using the Fermi statistics. This is, however, very difficult to control. Our solution has been to think of the effective potentials as interactions between new objects which are fermions with more structure than the original particles: we baptised them *quasiparticles.* They have more degrees of freedom and are related to the original particles, so that the Schwinger functions for the original particles can be computed from those of the quasiparticles by integrating them over the extra coordinates with suitable weights.

The redundance in degrees of freedom is reflected in a kind of *gauge symmetry* which we shall repeatedly exploit: the knowledge that the physical observables are necessarily expressible in terms of particle fields implies infinitely many identities and sum rules for the expressions involving the quasiparticles, quite analogous to the Ward identities of QED (e.g., see Sections 7, 11, and 12).

The remarkable property of the quasiparticles is that the effective potentials between the new quasiparticles seem to be bounded functions and furthermore the formalism allows us to take advantage of the oscillations of the integrands on a scale p_F^{-1} to show that also the integrations necessary to obtain the Schwinger functions of the quasiparticles can be performed and bounded uniformly. The divergence caused by the delta-function-like potentials, so troublesome in the formalism without the quasiparticles, does not cause problems because it is broken into a sum of many regular parts which can be controlled in the expressions of interest.

The fact that in the end we get control of an effective potential which, in a formalism without quasiparticles, would look divergent (but very short ranged) is not very surprising, since, intuitively, a residual delta-function-like potential (with bounded integral) is essentially equivalent to a zero potential because the delta interaction is trivial (a property valid if $d > 1$ and also, in the spinless case, if $d = 1$).

The $d = 1$ case is a borderline case: the integral of the deltalike function not only is bounded, but stays away from zero. This makes it harder to discuss the existence of the Fermi surface, particularly in the spinning case, as the delta interaction in one dimension may be nontrivial.

We arrived at the quasiparticle picture (see Section 5) by studying the cancellations due to the oscillations of the propagators on the Fermi length p_F^{-1} in two simple hierarchical models that we introduced: we do not reproduce here the labor performed on them because it would be a repetition of what we present. We nevertheless stress that even in this case, as already in the cases of the scalar field theories,[35] the analysis of a hierarchical model is very helpful and enlightening, and it has provided the essential key to this work.

We think that the interest of the notion of quasiparticle that we introduce goes beyond the technical aspect: it seems to be a precise mathematical notion which translates the phenomenological concept of quasiparticle originally due to Landau. They have strange properties which definitely distinguish them on a formal level from the Landau quasiparticles: they carry an intrinsic linear momentum (equal in size to the Fermi momentum p_F, i.e., they live on the Fermi surface) much as a spinning particle carries an intrinsic angular momentum. And they tend to join into pairs with opposite intrinsic linear momentum to the point that all our intuition is

based on the idea of calling *relevant* only the part of the effective potential representing the interaction between what we have called *Cooper pairs*. It is only a part of the pair potential.

As a final remark we wish to stress again that in the one-dimensional case there is no asymptotic freedom, because the effective potential does not tend to vanish, while in general dimension, as a potential between quasiparticles, it may go to zero, but not very fast (see Section 14). But in the quasiparticle language a potential which goes to a finite limiting value or even to zero too slowly corresponds, if interpreted as a pair potential between physical particles, to an approximate deltalike potential. Such a potential on scale h is an approximate delta function (see Appendix C) with width $2^h p_F^{-1}$, times a constant proportional to the size of the quasiparticle potential and to $2^{(d-1)h}$: hence its integral goes to 0, if $d>1$, as $h \to -\infty$, but this is *not* fast enough to prevent the effective potential between the physical particles from diverging essentially as $O(2^{-h})$ in *any dimension*. This shows that in any dimension the quasiparticle notion seems essential. Furthermore, in more than one dimension one can have an asymptotic freedom mechanism only if one represents the interaction as an interaction between quasiparticles.

The original preprint of this paper had a few lines missing where ref. 22 was introduced and comments on it were made: this was noticed by one of the referees, who asked in fact for more light on this point. We take the opportunity in the following lines to expand our original comments.

The theory of the one-dimensional case should be compared to the vast literature; see ref. 22. The comparison is not easy, as the levels of rigor demanded of renormalization group approaches have kept increasing steadily. In ref. 22 the theory of the Fermi gas is developed in great detail: one of the basic ideas appears to be the same guiding us in Section 15; namely, one tries to make use of exactly soluble models to understand the properties of others, nonexactly soluble (this[22] seems to be possible also in some models beyond the spinless case that we consider).

The problem of defining the beta function without approximations (like the *bandwidth cutoff* with the assumption of constancy or smoothness of the couplings within the band) and to all orders is not really considered and attacked in ref. 22; the calculations are confined to the lowest orders, with the major exception of Fowler's theorem (see p. 220 of ref. 22) and the related conjecture on the conservation under scaling of a suitable combination of coupling constants (which, however, can be regarded as constants only if the above approximations are considered). The analysis is made easier by neglecting completely the irrelevant terms and their contributions to the beta function. By contrast, the quantities that we call running couplings are, without approximations, constants (the identification of the

nonconstant part of the couplings used in ref. 22 with (*some*) of our irrelevant terms seems fairly clear). We do not neglect irrelevant terms and we study the beta function to all orders of perturbation theory, making estimates of the coefficients.

Our analysis can therefore be considered as an attempt at a more detailed understanding of the corrections that arise when one does not start with the approximations mentioned above and one does try to take into account the high orders and to put bounds on the beta function coefficients which are uniform in the cutoffs. In so doing we have been led to a precise, although apparently unconventional, notion of quasiparticles and to a more general theory (which is not restricted to one dimension) permitting the formulation of a renormalization group approach with a well-defined beta function [which, unfortunately, seems difficult to study even to second order (see Section 14) when $d>1$]. We have given an anomalous scaling interpretation of our one-dimensional results (going also through the analysis necessary to give a precise definition and bounds on an anomalous beta function): we plan to study the possible connection between the Fowler conjecture and our conjecture $G=0$, in the quest of an algebraic proof for it: our conjecture also relies on the exactly known properties of the Luttinger model, and the argument used by Fowler to formulate it seems to apply in our case as well. The explicit check (which is of some interest while looking for a general proof) of the vanishing of G to third order is being studied and in our context is more involved, if done by explicit calculation, because of the presence of the irrelevant contributions (which of course are *not small*).

1. SYMBOLS

Fermions in a periodic box $\Omega \subset R^d$, with side size L will be described in terms of creation and annihilation operators $\eta_{\mathbf{k}}^+$, $\eta_{\mathbf{k}}^-$, where $\mathbf{k}=(2\pi/L)\mathbf{n}$, $\mathbf{n}=(n_1,\ldots,n_d)\in Z^d$.

We choose units so that Planck's constant is $\hbar=1$ and introduce the following operators:

$$\begin{aligned}
&\psi_{\mathbf{x}}^{\pm} = L^{-d/2}\sum_{\mathbf{k}} e^{\pm i\mathbf{k}\cdot\mathbf{x}}\eta_{\mathbf{k}}^{\pm}\\
&T=\sum_{\mathbf{k}}\left(\frac{\mathbf{k}^2}{2m}-\mu\right)\eta_{\mathbf{k}}^{+}\eta_{\mathbf{k}}^{-}\equiv\int_{\Omega} d\mathbf{x}\left(\frac{1}{2m}\partial\psi_{\mathbf{x}}^{+}\,\partial\psi_{\mathbf{x}}^{-}-\mu\psi_{\mathbf{x}}^{+}\psi_{\mathbf{x}}^{-}\right)\\
&N=\sum_{\mathbf{k}}\eta_{\mathbf{k}}^{+}\eta_{\mathbf{k}}^{-}\equiv\int d\mathbf{x}\,\psi_{\mathbf{x}}^{+}\psi_{\mathbf{x}}^{-}\\
&V=\int d\mathbf{x}\,d\mathbf{y}\,\lambda_0(\mathbf{x}-\mathbf{y})\,\psi_{\mathbf{x}}^{+}\psi_{\mathbf{y}}^{+}\psi_{\mathbf{y}}^{-}\psi_{\mathbf{x}}^{-}
\end{aligned}\tag{1.1}$$

where μ, m will be fixed *a priori* and their combination $p_F = (2m\mu)^{1/2}$ will be called the *Fermi momentum*, while

$$e(\mathbf{k}) = \frac{\mathbf{k}^2}{2m} - \mu \equiv \frac{\mathbf{k}^2 - p_F^2}{2m} \tag{1.2}$$

will be called the *dispersion relation.*

We shall call H the *interaction Hamiltonian*:

$$H = T + V + \nu_0 N + \alpha_0 T \tag{1.3}$$

and we shall say that λ_0 is the *interaction potential*, ν_0 is the *chemical potential*, and α_0 is the *mass normalization.* Strictly speaking, according to the usual terminology of statistical mechanics, the chemical potential of the Hamiltonian (1.3) would be $\mu_0 = \mu(1+\alpha_0) - \nu_0$ and the particle mass would be $m_0 = m/(1+\alpha_0)$: hence ν_0 and α_0 are in fact related to the variations of the chemical potential and of the particle mass compared to the reference values μ and m, which are fixed *a priori.*

When $\lambda_0 = \alpha_0 = \nu_0 = 0$ the ground state of H is easy to find and it is simply given by

$$|F\rangle = \prod_{e(\mathbf{k})<0} \eta_{\mathbf{k}}^{+} |O\rangle \tag{1.4}$$

where $|O\rangle$ is the vacuum for the $\eta^{\pm}$ operators.

For Ω finite the ground-state properties as well as those of H can be deduced (obviously) from the *Schwinger functions*:

$$S(\mathbf{x}_1, t_1, \sigma_1, \ldots, \mathbf{x}_s, t_s, \sigma_s) = \frac{\operatorname{Tr} e^{-(\vartheta - t_1)H} \psi_{\mathbf{x}_1}^{\sigma_1} e^{-(t_1 - t_2)H} \psi_{\mathbf{x}_2}^{\sigma_2} \cdots \psi_{\mathbf{x}_s}^{\sigma_s} e^{-t_s H}}{\operatorname{Tr} e^{-\vartheta H}} \tag{1.5}$$

where $\vartheta \geqslant t_1 \geqslant \cdots \geqslant t_s \geqslant 0$, $\sigma_i = \pm$, and from the properties of such functions we can deduce the properties of the system at temperature ϑ^{-1}, too.

In the limit $\vartheta \to \infty$ the functions (1.5) can still be used to describe the properties of the ground state.

In this paper we plan to consider (1.5) and its limit as $\vartheta \to \infty$: if α_0, ν_0 are suitable functions of λ_0 and λ_0 is small enough, we show that (1.5) can be given a formal perturbation-theoretic meaning defining a formal expansion for the Schwinger functions of a state of the (infinite-volume) fermion system whose structure is described by the Landau picture of quasiparticles.

To impose Hamiltonian stability, we shall suppose that $\lambda_0 \geqslant 0$, i.e., a *repulsive* potential; we shall also suppose that the potential has short range, i.e., it decays rapidly at ∞, and that it is rotationally invariant.

2. CLASSICAL PERTURBATION THEORY

The following *time-dependent* fermion fields will be useful:

$$\psi^{\pm}_{\mathbf{x},t} = L^{-d/2} \sum_{\mathbf{k}} e^{\pm i\mathbf{kx} \pm e(\mathbf{k})t} \eta^{\pm}_{\mathbf{k}} \equiv e^{tT} \psi^{\pm}_{\mathbf{x}} e^{-tT} \tag{2.1}$$

They define the imaginary time fields.

Then, using the representation [where $V_0 \equiv V + \nu_0 N + \alpha_0 T$; see (1.3)]

$$e^{-tH} = \lim_{n \to \infty} \left[e^{-tT/n} \left(1 - \frac{tV_0}{n} \right) \right]^n \tag{2.2}$$

we find that the numerator of (1.5) becomes

$$\sum \pm \int \mathrm{Tr}\{e^{-\vartheta T} V_0(t'_1) \cdots V_0(t'_{p_1-1}) \psi^{\sigma_1}_{\mathbf{x}_1, t'_{p_1}} \cdots \psi^{\sigma_s}_{\mathbf{x}_s, t'_{p_1+\cdots+p_s}} \cdots V_0(t'_{p_1+\cdots+p_{s+1}})\} \, dt' \tag{2.3}$$

where $V_0(t) = e^{tT} V_0 e^{-tT}$ and the sum is over integers $p_1, p_2, \ldots$, while the integral is over all the t'_j variables with $j \neq p_1, p_1+p_2, \ldots, p_1+p_2+\cdots+p_s$; and $t'_{p_1}, t'_{p_1+p_2}, \ldots, t'_{p_1+p_2+\cdots+p_s}$ are fixed to be $t_1 > t_2 > \cdots > t_s \geqslant 0$, respectively; finally, the t' variables are constrained to decrease in their index j, and the sign $\pm$ is plus if the number of V_0 factors is even and minus otherwise.

Since the product of V_0's is an integral of a sum of products of $\psi^{\pm}_{\mathbf{x},t}$ operators and since the T is a quadratic Hamiltonian in the $\psi^{\pm}$ operators, Wick's theorem holds for evaluating $\mathrm{Tr}[\exp -\vartheta T(\cdot)]/\mathrm{Tr}(\exp -\vartheta T)$ and therefore it will be possible to express the various terms in (2.3) as suitable integrals of sums of products of expressions like

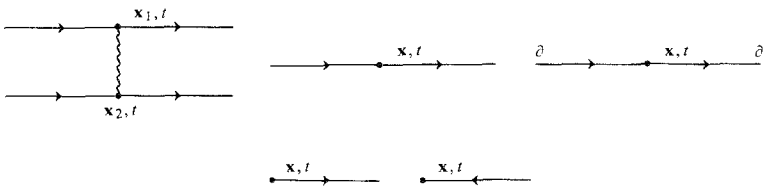
$$\begin{aligned} g_+(\xi, \tau) &= \mathrm{Tr}\, e^{-\vartheta T} \psi^-_{\mathbf{x},t} \psi^+_{\mathbf{x}',t'} / \mathrm{Tr}\, e^{-\vartheta T} \\ g_-(\xi, \tau) &= \mathrm{Tr}\, e^{-\vartheta T} \psi^+_{\mathbf{x},t} \psi^-_{\mathbf{x}',t'} / \mathrm{Tr}\, e^{-\vartheta T} \end{aligned} \tag{2.4}$$

if $\xi = \mathbf{x} - \mathbf{x}'$, $\tau = t - t' > 0$, which we combine to form a single function:

$$g(\xi, \tau) = \begin{cases} g_+(\xi, \tau) & \text{if } \tau > 0 \\ -g_-(-\xi, -\tau) & \text{if } \tau \leqslant 0 \end{cases} \tag{2.5}$$

Then it is easy to see, from Wick's theorem, that the generic term in (2.3) can be expressed graphically as follows.

One lays down graph elements like



(2.6)

symbolizing, respectively,

$$-\lambda_0(\mathbf{x}_1-\mathbf{x}_2)\,\psi^+_{\mathbf{x}_1,t}\psi^+_{\mathbf{x}_2,t}\psi^-_{\mathbf{x}_2,t}\psi^-_{\mathbf{x}_1,t}$$

$$-v_0\psi^+_{\mathbf{x},t}\psi^-_{\mathbf{x},t}$$

$$(\alpha_0/2m)(i\partial+\mathbf{p}_F)\,\psi^+_{\mathbf{x},t}(i\partial+\mathbf{p}_F)\,\psi^-_{\mathbf{x},t}$$

$$\psi^+_{\mathbf{x},t}$$

$$\psi^-_{\mathbf{x},t}$$

with $\mathbf{p}_F$ being a vector of size p_F (of any pre-fixed direction). We call the graph elements in (2.6) 1, 2, 3, and 4 (the last representing the final two elements), respectively.

One should then draw $n+s$ such elements so that the first n have a shape of the form 1, 2 or 3 with labels $(\mathbf{x}, t)$ attached arbitrarily to the vertices ("free labels") and the last s have a shape of the form 4 (representing, respectively, $\psi^-_{\mathbf{x},t}$ or $\psi^+_{\mathbf{x},t}$) and carry "external labels" $(\mathbf{x}_1, t_1),\ldots, (\mathbf{x}_s, t_s)$.

Then one considers all possible ways of joining together lines in pairs so that no line is left over unpaired and only lines with consistent orientations are allowed to form a pair.

To each graph we assign a sign $\sigma = \pm$ obtained by considering the permutation necessary to bring next to each other the pairs of operators which in the given graph are paired (one says also *contracted*), with the ψ^- to the left of the associated ψ^+, and then setting $\sigma = (-1)^\pi$ if π is the permutation parity.

To each graph we assign a *value* which is the integral over the free vertices of the product of the sign factor times the product of factors $g(\xi, \tau)$ (or of some of its derivatives) for every line λ with an arrow pointing from $(\mathbf{x}_1, t_1)$ to $(\mathbf{x}_2, t_2)$ with $\xi = (\mathbf{x}_2 - \mathbf{x}_1)$, $\tau = t_2 - t_1$, times a factor $-\lambda_0(\mathbf{x}_1 - \mathbf{x}_2)$ for every wiggly line joining $(\mathbf{x}_1, t)$ to $(\mathbf{x}_2, t)$, times a factor $-v_0$ or $\alpha_0/2m$ for every vertex of the type with only two lines.

The *propagator* function g is given by (2.5) and can be represented as

$$g(\xi,\tau)=L^{-d}\sum_{\mathbf{k}} e^{-i\mathbf{k}\xi}\left\{\frac{e^{-\tau e(\mathbf{k})}}{1+e^{-\vartheta e(\mathbf{k})}}\chi(\tau>0)-\frac{e^{-(\vartheta+\tau)e(\mathbf{k})}}{1+e^{-\vartheta e(\mathbf{k})}}\chi(\tau<0)\right\}$$

$$\xrightarrow[\vartheta\to\infty]{L\to\infty}\frac{1}{(2\pi)^d}\int d^d\mathbf{k}\, e^{-\tau e(\mathbf{k})}e^{-i\mathbf{k}\xi}(\chi(e(\mathbf{k})>0)\,\chi(\tau>0)$$

$$-\chi(e(\mathbf{k})<0)\,\chi(\tau\leqslant 0))$$

$$=\frac{1}{(2\pi)^{d+1}}\int dk_0\, d^d\mathbf{k}\,\frac{e^{-i(k_0(\tau+0^-)+\mathbf{k}\xi)}}{-ik_0+e(\mathbf{k})} \tag{2.7}$$

where χ ("condition") $=1$ if "condition" is satisfied and $\chi=0$ otherwise. The reason why $\tau=0$ requires the use of g^- is simply that such a case can only arise in the anomalous graphs in which a pairing occurs between lines emerging from points with the same time index. Disregarding sets of times of measure zero in the integral (2.3), such pairings can only arise by pairing lines representing operators ψ^+, ψ^- in the same V_0 factor of (2.3); therefore ψ^+ is always to the left of ψ^- and the propagator is necessarily $g_-(\cdot,0)$.

The sum of the graph values over all graphs yields the expansion for (2.3) up to a multiplying factor $\mathrm{Tr}\, e^{-\vartheta T}$.

3. EUCLIDEAN FORMALISM

The numerator and denominator of (1.5) admit a concise representation as a formal functional integral.

For this purpose one introduces an algebra constructed from symbols ψ^+_ξ, ψ^-_η, with $\xi,\eta\in R^{d+1}$. The symbols are considered to form a basis out of which the algebra is constructed by assuming that the $\psi^\pm_\xi$ satisfy the following anticommutation identities:

$$\{\psi^+_\xi,\psi^+_\eta\}=0,\qquad \{\psi^-_\xi,\psi^-_\eta\}=0,\qquad \{\psi^+_\xi,\psi^-_\eta\}=0 \tag{3.1}$$

Following standard practice, the symbols $\psi^\pm$ are now used to denote new objects, which should not be confused with the fermion operators of the previous sections.[10]

The *integration* $P(d\psi)$ is simply defined by assigning the value of the integral of a monomial:

$$\int P(d\psi)\,\psi^{\sigma_1}_{\xi_1}\cdots\psi^{\sigma_n}_{\xi_n} \tag{3.2}$$

By definition, the value of such an integral is given by Wick's rule. One considers all the *pairings*, i.e., all possible ways of collecting in pairs the n symbols ψ_ξ^σ in such a manner that no symbols appear in more than one pair and the two symbols of each pair have a different σ: calling π the parity of the permutation on $\xi_1, \ldots, \xi_n$ necessary to put the ψ's of each pair next to each other with the ψ^- to the left, then one assigns to each pairing a value which is $(-1)^\pi$ times the product of factors $g(\xi - \xi')$ for each pair formed by pairing ψ_ξ^- with $\psi_{\xi'}^+$; the value of (3.2) is the sum over the values of all the pairings.

From the analysis of the previous section it should be clear that the numerator in (1.5), divided by $\operatorname{Tr} e^{-\beta T}$, is given by

$$\int P(d\psi) \prod_{i=1}^{n} (\psi_{\mathbf{x}_i, t_i}^{\sigma_i}) \exp -\left\{ \int_{\Omega^2 \times [0,\beta]} \lambda_0(\xi - \eta)\, \psi_{\xi,t}^+ \psi_{\eta,t}^+ \psi_{\eta,t}^- \psi_{\xi,t}^-\, d\xi\, d\eta\, dt \right.$$

$$\left. + \int_{\Omega \times [0,\beta]} \left[\nu_0 \psi_\xi^+ \psi_\xi^- + \alpha_0 \left(\frac{1}{2m} \partial\psi_\xi^+ \, \partial\psi_\xi^- - \mu \psi_\xi^+ \psi_\xi^- \right) \right] d\xi \right\} \tag{3.3}$$

and (3.3) has to be interpreted in the sense of formal power series in λ_0, ν_0, α_0.

The abstract objects ψ_ξ^σ are called *Grassmanian variables* or *Euclidean fermion fields*, and the linear form defined by (3.2) on such an algebra is called an *anticommutative Gaussian process*. The theory of the integrals (3.3) is equivalent to the problem of giving a meaning to the formal perturbation series of Section 2.

The analysis of the Schwinger functions is not very convenient. It is more convenient to work with the *truncated* Schwinger functions. The latter are related to the Schwinger functions by finite algebraic relations. The formal definition is in terms of auxiliary Grassmanian variables $\varepsilon_x^\pm$, anticommuting also with the $\psi^\pm$ fields. One sets

$$S^T(x_1\sigma_1, \ldots, x_s\sigma_s)$$

$$= \frac{\delta^s}{\delta\varepsilon_{x_1}^{\sigma_1} \cdots \delta\varepsilon_{x_s}^{\sigma_s}} \log \int P(d\psi) \exp\left[-V_0(\psi) + \int dx\, (\varepsilon_x^+ \psi_x^- + \psi_x^+ \varepsilon_x^-) \right] \Big|_{\varepsilon = 0} \tag{3.4}$$

where δ denotes the formal functional derivative, which, together with the logarithm and exponential, is defined in the sense of formal power series. Note that, however, $S^T(x+, y-) \equiv S(x-, y+)$, and we denote $S(x-, y+) \equiv S(x-y)$, as it plays a special role.

A very convenient object is the generating function of the truncated Schwinger functions:

$$\mathscr{S}(\varepsilon)=\sum_{s=2}^{\infty}\sum_{\sigma_1\cdots\sigma_s}\int\frac{dx_1\cdots dx_s}{s!}S^T(x_1\sigma_1,\ldots,x_s\sigma_s)\,\varepsilon^{\sigma_1}_{x_1}\cdots\varepsilon^{\sigma_s}_{x_s} \tag{3.5}$$

which is related to the notion of *effective potential* defined by

$$e^{-V_{\rm eff}(\varepsilon)}=\int P(d\psi)\,e^{-V_0(\psi+\varepsilon)} \tag{3.6}$$

The relation is, if $(g\varepsilon)^-=g*\varepsilon^-$ and $(g\varepsilon)^+=\varepsilon^+*g'$, where the $*$ denotes convolution, g is the propagator in (2.7), and $g'(x)=g(-x)$, the following:

$$-V_{\rm eff}(g\varepsilon)+(\varepsilon^+,\,g\varepsilon^-)=\mathscr{S}(\varepsilon) \tag{3.7}$$

(we learnt, in the scalar fields case, the above relation from L. Rosen).

The above relations are formally trivial if one treats $\int P(d\psi)\cdot$ as an ordinary integral with respect to a measure proportional to

$$d\psi^+\,d\psi^-\exp\left\{-\int[\psi^+_x(\partial_t+(-\Delta+p_{\rm F}^2)/2m]\,\psi^-_x\,dx\right\} \tag{3.8}$$

and proceeding to the *change of variables* $\psi+g\varepsilon=\tilde{\psi}$.

Of course the formal argument is meaningless as presented; however, if one writes the above calculations (i.e., the change of variables) as relations between formal power series in the fermion fields, one sees that they are indeed valid.

We can express $V_{\rm eff}(\varepsilon)$ as a series like (3.5), thus defining kernels $V_{\rm eff}(x_1\sigma_1,\ldots,x_s\sigma_s)$, and among them the $V_{\rm eff}(z-,z'+)\equiv V_{\rm eff}(z-z')$ will play a special role.

In fact we see that the theory of the effective potential is equivalent to the theory of the two-point Schwinger function:

$$S(x-y)=g(x-y)-\int g(x-z)\,V_{\rm eff}(z-z')\,g(z'-y)\,dz\,dz' \tag{3.9}$$

and the property (0.2) can be translated into properties of the Fourier transform $\hat{V}_{\rm eff}(k)$ of the effective potential. The properties are:

1. The effective potential has short range: i.e., $\hat{V}_{\rm eff}$ is a smooth function in k.

2. The effective potential vanishes on the Fermi surface:

$$\hat{V}(k_0,\mathbf{k})=0\qquad\text{if}\quad k_0=0,\quad|\mathbf{k}|=p_{\rm F} \tag{3.10}$$

3. The expansion of $\hat{V}(k)$ around $k_0 = 0$, $|\mathbf{k}| = p_F$ has the form

$$\hat{V}_{\text{eff}}(k_0, \mathbf{k}) = \bar{\alpha} \frac{\mathbf{k}^2 - p_F^2}{2m} - \bar{\zeta} i k_0 + \cdots \tag{3.11}$$

4. The coefficients $\bar{\alpha}$ and $\bar{\zeta}$ are equal: $\bar{\alpha} = \bar{\zeta} = \zeta_\infty$.

The above four conditions imply that

$$S(k_0, \mathbf{k}) = \frac{1}{-ik_0 + e(\mathbf{k})} - \frac{\zeta_\infty(-ik_0 + e(\mathbf{k}))}{[-ik_0 + e(\mathbf{k})][-ik_0 + e(\mathbf{k})]} + \cdots \tag{3.12}$$

i.e., (0.2) holds with $Z^{-1} = 1 - \zeta_\infty$ in a sense which depends on how good a control one has on the regularity properties of the function $\hat{V}_{\text{eff}}(k)$ near $k_0 = 0$, $|\mathbf{k}| = p_F$ (i.e., on the Fermi surface).

In perturbation theory one tries to find expansions for $\bar{\alpha}$, $\bar{\zeta}$ and for the value $\bar{v} = V_{\text{eff}}(0, p_F)$ in powers of λ_0, α_0, v_0. The idea is that, given p_F, m and given λ_0 small enough, one can express the parameters $\bar{v}$, $\bar{\alpha}$, $\bar{\zeta}$ and impose the conditions

$$\bar{v}(\alpha_0, v_0, \lambda_0) = 0, \qquad \bar{\alpha}(\alpha_0, v_0, \lambda_0) = \bar{\zeta}(\alpha_0, v_0, \lambda_0) \tag{3.13}$$

to fix the two free parameters α_0, v_0 (i.e., the bare mass and the bare chemical potential).

It is nontrivial and really remarkable that this can be achieved formally to all orders of perturbation theory in a sense which is not literally the above (which is incorrect), but in fact much better: the expansion is possible if $\bar{v}(\alpha_0, v_0, \lambda_0)$ is not really constructed as a function of α_0, v_0, λ_0 for all their values near 0 (say). Rather, one uses only the parameters α_0, λ_0 and determines v_0 as a power series in α_0, λ_0 so that the coefficients of the formal expansion of $\bar{v}$ in powers of α_0, λ_0 vanish. In other words, one only defines $\bar{v}$ for the values of α_0, v_0, λ_0 for which its value is 0. One finds that this is possible formally to all orders and determines uniquely formal series for $v_0(\alpha_0, \lambda_0)$, $\bar{\alpha}(\alpha_0, \lambda_0)$, and $\bar{\zeta}(\alpha_0, \lambda_0)$; one imposes subsequently that $\bar{\alpha} = \bar{\zeta}$ (which can be solved in the sense of formal power series to all orders).

This shows that the perturbation series is possible *only* if the chemical potential is fixed at the *right value*. The similarity with the KAM theory of perturbations of Hamiltonian quasiperiodic motions is here striking.

The work on the above perturbation theory is classical; see refs. 1–7 and in particular refs. 8 and 9. The results follow again as a consequence of our work, which also provides bounds for the size of the coefficients of the expansion.

However, the mentioned perturbation theory is unsatisfactory, as pointed out in refs. 11–13 and 15.

In fact, one can easily identify special classes of contributions to the

functions $\bar{\alpha}(\alpha_0, \lambda_0)$, $\bar{\zeta}(\alpha_0, \lambda_0)$, $\bar{v}_0(\alpha_0, \lambda_0)$, given by integrals of sums of geometric series: if summed formally, they become divergent, although the term-by-term integrals are convergent. The mechanism is the same as that of the integral

$$\sum_{n=0}^{\infty} \int_0^1 \beta^n \lambda_0 (\lambda_0 \log |x|^{-1})^n \, dx = \int_0^1 \frac{\lambda_0}{1 + \beta \lambda_0 \log |x|^{-1}} \, dx \tag{3.14}$$

i.e., we see that the integrals are finite order by order, but their formal sum diverges at least if $\beta\lambda_0 < 0$. This shows that a correct perturbation theory, even if only formal, cannot be confined to the proof that one has finiteness order by order.[5]

On the other hand, the above situation is typical of the renormalization group approaches. We expect that there should be a notion of *running form factors*, which we denote $\mathbf{v}_h = (\lambda_h, v_h, \alpha_h, \zeta_h)$, and that all the interesting quantities should be expandible in a formal power series in $\mathbf{v}_h$. The power series is possibly even convergent if $|\mathbf{v}_h|$ is small enough for all h. Then the problem becomes that of controlling the dependence of $\mathbf{v}_h$ on h as functions of the parameters of the theory $(\alpha_0, v_0, \lambda_0)$: one would like to prove that they stay uniformly bounded and small, at least if λ_0 is chosen small enough and the others conveniently.

4. EFFECTIVE POTENTIALS

The basic tool to produce estimates on the graphs of perturbation theory is the *multiscale decomposition* of the Euclidean fermion fields. Fixing arbitrarily a momentum scale p_0, it is generated by the following identities, starting from (2.7):

$$\begin{aligned}
g(x) &= \int \frac{dk_0 \, d^d\mathbf{k}}{(2\pi)^{d+1}} \frac{\exp[i(-k_0 t - \mathbf{kx})]}{-ik_0 + e(\mathbf{k})} \\
&= \int \frac{dk_0 \, d^d\mathbf{k}}{(2\pi)^{d+1}} \frac{\exp[i(-k_0 t - \mathbf{kx})]}{k_0^2 + e(\mathbf{k})^2} [ik_0 + e(\mathbf{k})] \\
&= \sum_{-\infty}^{0} \int_{p_0^{-2} 2^{-2n-2}}^{p_0^{-2} 2^{-2n}} d\alpha \int \frac{dk_0 \, d^d\mathbf{k}}{(2\pi)^{d+1}} \\
&\quad \times (\exp\{i(-k_0 t - \mathbf{kx}) - \alpha[k_0^2 + e(\mathbf{k})^2]\})[ik_0 + e(\mathbf{k})] \\
&\quad + \int_0^{p_0^{-2}/4} d\alpha \int \frac{dk_0 \, d^d\mathbf{k}}{(2\pi)^{d+1}} \\
&\quad \times (\exp\{i(-k_0 t - \mathbf{kx}) - \alpha[k_0^2 + e(\mathbf{k})^2]\})[ik_0 + e(\mathbf{k})] \\
&\equiv \sum_{-\infty}^{1} \bar{g}_n(x)
\end{aligned} \tag{4.1}$$

where $x=(\mathbf{x}, t)$ and the functions $\bar{g}_n(x)$ have simple scaling properties, if $n \leqslant 0$. In fact,

$$\begin{aligned} \bar{g}_n(\mathbf{x}, t) &= 2^n g_n^s(\xi, \tau) \frac{\sin p_F |\mathbf{x}|}{p_F |\mathbf{x}|} + 2^{2n} \frac{p_0}{p_F} g_n^c(\xi, \tau) \cos p_F |\mathbf{x}|, \qquad d=3 \\ \bar{g}_n(x, t) &= 2^n g_n^s(\xi, \tau) \cos p_F x - 2^n g_n^c(\xi, \tau) \sin p_F x, \qquad d=1 \end{aligned} \tag{4.2}$$

where $\xi = 2^n p_0 \mathbf{x}$, $\tau = 2^n p_0 t$, and, if $n \leqslant 0$,

$$|\partial_\tau^p \partial_\xi^{\mathbf{r}} g_n^\alpha(\xi, \tau)| \leqslant 2^{n(p+|\mathbf{r}|)} G_\beta(\xi, \tau)\, p_0^{p+|\mathbf{r}|+1} p_F^{d-1}, \qquad \alpha = s, c \tag{4.3}$$

where $\beta = p_F/m$ and G_β, which depends on p, $\mathbf{r}$, q but not on n, is uniformly bounded and decays at ∞ faster than any power of its arguments.

Furthermore, g^s, g^c are holomorphic in ξ, τ and admit a bound with G_β decaying exponentially fast at ∞ for ξ_i, τ in a complex plane strip of pre-fixed size, and with a decay rate bounded by a quantity that also can be arbitrarily pre-fixed.

See Appendix A for our choice of g^s, g^c and for a detailed check of the above statements.

We shall often choose units so that $\beta =$ velocity at the Fermi surface $= 1$ and $\hbar = 1$: we call such units *natural* and in such units the dimension of λ is an inverse length.

A convenient and natural choice for p_0 is to fix $p_0^{-1} =$ range of the interaction potential: the latter will be supposed to have the form $\lambda(\mathbf{x}) = p_0^d \bar{\lambda}(p_0 \mathbf{x})$ with $\bar{\lambda}$ having range 1.

We can use (4.2) to represent our Euclidean fermion fields as sums of other independent Grassmanian variables:

$$\psi_{\mathbf{x},t}^\sigma = \sum_{n=-\infty}^{1} \int \psi_{\mathbf{x},t,\boldsymbol{\omega}}^{(n)\sigma} e^{ip_F \sigma \boldsymbol{\omega} \cdot \mathbf{x}}\, d\boldsymbol{\omega} \tag{4.4}$$

where the propagator between the fields $\psi_{\mathbf{x},t,\boldsymbol{\omega}}^{(n)-}$ and $\psi_{\mathbf{x}',t',\boldsymbol{\omega}'}^{(n)+}$ is given, if $\xi = (\mathbf{x}-\mathbf{x}')\, p_0 2^n$ and $\tau = (t-t')\, 2^n p_0$, by

$$\begin{aligned} &2^n \delta(\boldsymbol{\omega}-\boldsymbol{\omega}')\, g_n(\xi, \tau, \boldsymbol{\omega}) \\ &\equiv \delta(\boldsymbol{\omega}-\boldsymbol{\omega}') \sum_{\alpha=c,s} 2^n \left[\left(2^n \frac{p_0}{p_F} \right)^{\varepsilon(\alpha)} - \varepsilon(\alpha)\, i\boldsymbol{\omega}\xi \right] g_n^\alpha(\xi, \tau), \qquad d=3 \\ &2^n \delta(\boldsymbol{\omega}-\boldsymbol{\omega}')\, g_n(\xi, \tau, \boldsymbol{\omega}) \equiv \delta(\boldsymbol{\omega}-\boldsymbol{\omega}')\, 2^n [g_n^s(\xi, \tau) - i\omega g_n^c(\xi, \tau)], \qquad d=1 \end{aligned} \tag{4.5}$$

with $\varepsilon(c) = 1$ and $\varepsilon(s) = 0$, and δ is the delta function relative to the measure $d\boldsymbol{\omega}$ defined by

$$d\boldsymbol{\omega} = \frac{\delta(|\boldsymbol{\omega}|-1)}{\Omega_d\, |\boldsymbol{\omega}|^{d-1}}\, d^d\boldsymbol{\omega} \tag{4.6}$$

where Ω_d is the surface of the d-dimensional sphere, i.e., δ is defined on a test function f by

$$\int \delta(\omega - \omega')\, d\omega'\, f(\omega') \equiv f(\omega) \tag{4.7}$$

If $n=1$, the fields $\psi^{(1)}$ are given an *indecomposed propagator*: (4.2) still holds, but (4.3) holds only outside a neighborhood of the origin (because there is no ultraviolet cutoff in g_1).

Formula (4.5) can be read in the jargon of the renormalization group as saying that the dimension of the fields $\psi^{\pm}_{x\omega}$ is 1/2, independently of d.

The d independence of (4.5) reflects the fact that the particle propagator (4.1) has a singularity on the Fermi surface which has codimension 2 [1 (spatial) + 1 (temporal)]. The asymptotic scaling and the symmetry between space and time in the quasiparticle propagator reflects the nature of the singularity in the radial direction, linear in k_0 and $|\mathbf{k}| - p_F$ (the latter is the variable called h in Appendix A).

In checking (4.4) and (4.5), one uses, for $d=3$, the identities

$$\frac{\sin|\mathbf{x}|}{|\mathbf{x}|} = \int d\omega\, e^{-i\omega\cdot\mathbf{x}}, \qquad \cos|\mathbf{x}| = \int (1 - i\omega\mathbf{x})\, d\omega\, e^{-i\omega\cdot\mathbf{x}} \tag{4.8}$$

and similar ones if $d=1$.

Asymptotically, as $n \to -\infty$ one has, for $d=1, 3$ [see (A.9)],

$$g_n(\xi, \tau, \omega) = C_d(\tau - i\beta^{-1}\omega\xi)\, \gamma_0(\tau^2 + \beta^{-2}\xi^2) \tag{4.9}$$

where [see (A.10)]

$$\gamma_0(z) = \int_{1/4}^{1} e^{-z/4\alpha} \frac{d\alpha}{\alpha^2}, \qquad C_d = \frac{\pi\Omega_d\, p_F^{d-1} p_0}{2(2\pi)^{d+1}\beta} \tag{4.10}$$

It appears that the fields $\psi^{(n)}_{\mathbf{x},t,\omega}$ are naturally associated with the length scale $2^{-n}p_0^{-1}$.

The following notation will be used:

$$\psi^{(\leq N)\sigma}_{\xi} \equiv \sum_{n=-\infty}^{N} \psi^{(n)\sigma}_{\xi} \tag{4.11}$$

and, if $\Lambda = \Omega \times [-T, T]$, we define the effective potential $V^{(N)}$ on scale $p_0^{-1}2^{-N}$ with ultraviolet cutoff $U \geq 0$ as

$$\exp -V^{(N)}(\psi^{(\leqslant N)})$$

$$= \int P(d\psi^{(N+1)}) \cdots P(d\psi^{(U)})$$

$$\times \exp -\Big\{ \int_A \lambda_0(\xi - \eta)\, \psi^{(\leqslant U)+}_{\xi,t} \psi^{(\leqslant U)-}_{\xi,t} \psi^{(\leqslant U)+}_{\eta,t} \psi^{(\leqslant U)-}_{\eta,t}\, d\xi\, d\eta\, dt$$

$$+ \int_A v_0 \psi^{(\leqslant U)+}_{\xi} \psi^{(\leqslant U)-}_{\xi}\, d\xi + \int_A \frac{\alpha_0}{2m} \psi^{(\leqslant U)+}_{\xi} (-\partial_\xi^2 - p_F^2)\, \psi^{(\leqslant U)-}_{\xi}\, d\xi$$

$$+ \int_A \zeta_0 \psi^{(\leqslant U)+}_{\xi} \partial_t \psi^{(\leqslant U)-}_{\xi}\, d\xi \Big\} \tag{4.12}$$

where λ_0 will be called the *potential*, v_0 the *chemical potential*, α_0 the *mass normalization*, and ζ_0 the *discontinuity parameter*.

Eventually one shall only be interested in the choices of v_0, α_0 permitting one to interpret the state of the Fermi system as a state with a Fermi surface at p_F (*a priori* prescribed) on which Landau quasiparticles move with a given mass m, and one wishes to take $\zeta_0 = 0$ (otherwise one would lose the physical interpretation of the theory).

The propagators of the various fields obtained from the fields ψ^n in (4.4) or (4.11) by taking summations over n between various extremes will be denoted by appending to g or $\bar{g}$ subscripts like $(\leqslant h)$ if the extremes are $(-\infty, h)$, or $(\geqslant h)$ if the extremes are $(h, 0)$, or (h, h') if the extremes are both finite.

As mentioned in the introduction, we shall fix the ultraviolet cutoff $U = 0$, i.e., we fix it at scale p_0^{-1}. The effective potential on scale h generates the truncated Schwinger functions $S_{(\geqslant h)}$ of a theory with infrared cutoff at scale $p_0^{-1} 2^{-h}$.

In the free case the evaluation of the function $S_{(\geqslant h)}(x, y)$ at points $2^{-h}x_0$, $2^{-h}y_0$ (i.e., on scale $p_0^{-1}2^{-h}$) leads to (fixing $d = 3$ as an example)

$$S^0_{(\geqslant h)}(x_0, y_0) = 2^{-2h} S_{(\geqslant h)}(2^{-h}(x_0 - y_0))$$

$$= \sin(p_F 2^{-h} |\mathbf{x}_0 - \mathbf{y}_0|)\, g^s_{(\geqslant h, \text{scaling})}(x_0 - y_0)$$

$$+ \cos(p_F 2^{-h} |\mathbf{x}_0 - \mathbf{y}_0|)\, g^c_{(\geqslant h, \text{scaling})}(x_0 - y_0) \tag{4.13}$$

where, if $\sigma = s, c$, we have defined

$$g^\sigma_{(\geqslant h, \text{scaling})}(x) = \sum_{n=h}^{U} 2^{2(n-h)} g^\sigma_n(2^{n-h} p_0 x)$$

$$\xrightarrow[h \to \infty]{} \sum_{q=0}^{+\infty} 2^{2q} g^\sigma_\infty(2^q p_0 x)$$

$$\equiv g^\sigma_{\text{scaling}}(x) \tag{4.14}$$

so that [see the asymptotic formulas in Appendix A, (A6)–(A10)]

$$g^{\sigma}_{\text{scaling}}(x) = \frac{\Omega_3 \pi p_F}{2(2\pi)^4 \beta} \gamma_{\text{scaling}}(x) \begin{cases} t/|\mathbf{x}|, & \sigma = s \\ \beta^{-1}, & \sigma = c \end{cases}$$

$$\gamma_{\text{scaling}}(x) = \sum_{q=0}^{+\infty} 2^{2q} p_0^2 \gamma_0(2^q p_0 x) \tag{4.15}$$

$$= 4 \frac{\exp(-p_0^2 x^2/4)}{x^2} \qquad \text{if} \quad x^2 = t^2 + \mathbf{x}^2 \beta^{-2}$$

Hence we see that if x^2 is fixed ($=1$, say), we can read from the leading behavior of $S^0_{(\geqslant h)}(x)$ as $h \to -\infty$ the value of the Fermi radius p_F and the particle mass m: the first is directly given by the period of the oscillations in space (i.e., $2\pi 2^h/p_F$ in our rescaled units) [see (4.13)], and the second is deduced from β ($\beta = p_F/m$) [see (4.15)].

The above analysis shows the possibility of defining the Fermi surface and the particle mass via the analysis of the effective potential on scale h: using (3.9), one could require $V^{(h)}_{\text{eff}}$ to have the properties necessary to imply that the $S^T_{(\geqslant h)}(x, y)$, defined by the rhs of (3.9), indeed have the asymptotic properties described by (4.13)–(4.15).

This is not, however, our choice: in our opinion the Fermi surface and the mass are more naturally associated with the new concept of quasiparticles that we introduce in the next section and that we identify with the well-known phenomenological notion of quasiparticle introduced by Landau (see ref. 15).

5. QUASIPARTICLES. MEAN FIELD THEORY AND GAP EQUATION. DEFINITION OF FERMI SURFACE

We now come to the problem of defining what we shall adopt as the primary definition of Fermi surface and quasiparticle mass.

The previous formalism suggests a radical change in point of view. The case $d=1$ is clearest and we treat it first; we keep, however, the general notation for later reference (hence, if $d=1$, the $\int d\omega$ will be the average over $\omega = \pm 1$, etc.).

We imagine a system of fermions characterized by an *intrinsic linear momentum* $p_F \omega$ and external linear momentum $\mathbf{k}$. The intrinsic linear momentum should be thought of as the linear momentum analogue of the spin. Such particles will be described by fermion fields $\psi^{\pm}_{\mathbf{x},t,\omega}$ with propagator $\delta(\omega - \omega')\, g(\mathbf{x} - \mathbf{x}', t - t', \omega)$, where

$$g(\mathbf{x}, t, \omega) = \int \frac{e^{-i(\mathbf{k}\mathbf{x} + k_0 t)}}{\varepsilon(k, \omega)} \frac{d^{d+1}k}{(2\pi)^{d+1}} \tag{5.1}$$

The new fermion variables may be used to represent the fermion variables of the previous sections as

$$\psi^{\pm}_{\mathbf{x},t} = \int d\boldsymbol{\omega}\, e^{\pm ip_F \boldsymbol{\omega}\mathbf{x}} \psi^{\pm}_{\mathbf{x},t,\boldsymbol{\omega}} \tag{5.2}$$

The function $\varepsilon(k, \boldsymbol{\omega})$ has to be such that

$$g(\mathbf{x}, t) = \int \frac{d^{d+1}k}{(2\pi)^{d+1}} \frac{e^{-i(k_0 t + \mathbf{kx})}}{-ik_0 + (\mathbf{k}^2 - p_F^2)/2m} \equiv \int d\boldsymbol{\omega}\, e^{-ip_F \boldsymbol{\omega}\mathbf{x}} g(\mathbf{x}, t, \boldsymbol{\omega}) \tag{5.3}$$

Keeping the restriction $d=1$, an elementary calculation shows that $\varepsilon(k, \omega) = -ik_0 + \beta\boldsymbol{\omega}\mathbf{k} + O(\mathbf{k}^2)$ as a consequence of (A6)–(A10). It is easy to check that there are ∞-many choices of $\varepsilon(k, \boldsymbol{\omega})$. Every choice of the function χ in Appendix A, for instance, provides a different $\varepsilon(k, \boldsymbol{\omega})$ still satisfying $\varepsilon(k, \boldsymbol{\omega}) = -ik_0 + \beta\boldsymbol{\omega}\mathbf{k} + O(k^2)$. All choices *agree* to first order near the Fermi surface $k=0$. We shall fix our choice by using the propagators of Appendix A.

This means that the free fermion system in a ground state with Fermi momentum at p_F can be considered as a system of *quasiparticles* in the vacuum carrying an intrinsic linear momentum equal to a Fermi sphere momentum. The *dispersion relation* is *almost* linear in the sense that the system on large scales, i.e., $\mathbf{k}$ small, will show a dispersion relation essentially identical to $\eta(\mathbf{k}) = \boldsymbol{\omega}\mathbf{k}\beta$: this property seems to remain valid even in the presence of interaction and this is the main result of the analysis of the coming sections, which are developed by letting intuition be led by the idea that the *quasiparticles* are to be taken seriously.

The case $d=3$ is discussed similarly, but for reasons that are made clear below, we only consider it when we have an ultraviolet cutoff at p_0. The propagator $g(\mathbf{x}-\mathbf{x}', t-t', \boldsymbol{\omega})$ will be therefore defined as [see (4.5), (4.6)]

$$g(\mathbf{x}, t, \boldsymbol{\omega}) = \sum_{h=-\infty}^{0} 2^h g_h(\mathbf{x}, t, \boldsymbol{\omega}) \tag{5.4}$$

Using the work of Appendix A, one easily checks that (5.4) can be put in the form (5.1) by setting

$$\varepsilon(k, \boldsymbol{\omega}) = -ik_0 a(k) + b(k) + \beta\boldsymbol{\omega}\cdot\mathbf{k}[c(k) - ik_0\, d(k)]$$

with a, b, c, and d functions of $k_0^2 + \beta^2\mathbf{k}^2$ growing very fast as $k \to \infty$ (exponentially, because of the ultraviolet cutoff). The difference between the $d=1$ case and the $d>1$ cases is not so much in the introduction of the ultraviolet cutoff (which could be easily avoided by extending the sum over

h up to 1), but rather in the fact that the $\varepsilon(k, \omega)$, even in the absence of an ultraviolet cutoff, would have to be really different from $-ik_0 + \beta\omega \cdot \mathbf{k} + O(\mathbf{k}^2)$. The main feature of $\varepsilon(k, \omega)$, common to $d = 1, 3$, is that $1/|\eta|$, with $\eta = \mathrm{Re}\, \varepsilon$, is not integrable at 0 over k_0, $\mathbf{k}$; it is, however, integrable at ∞ with respect to $\mathbf{k}$ (but, if $d = 3$, only because of the ultraviolet cutoff).

The leading singularity at $k = 0$ is easy to evaluate from Appendix A, (A7)–(A19), and it is

$$\varepsilon(k, \omega) = [-ik_0 + \beta\omega \cdot \mathbf{k})(k_0^2 + \beta^2(\omega \cdot \mathbf{k})^2]^{-1} (k_0^2 + \beta^2\mathbf{k}^2)^2 (2\beta)^{-2} p_F^{-2}$$

We can generalize our previous problem into that of studying Euclidean Fermi fields with interaction

$$\begin{aligned} V^{(0)} = & \int d\mathbf{x}\, dt\, d\omega_1\, d\omega_2 \exp[ip_F(\omega_1 - \omega_2)\mathbf{x}] \\ & \times \psi^+_{\mathbf{x}, t, \omega_1} \left(v_0 + \frac{\alpha_0 p_F^2}{2m} (2ip_F^{-1}\omega_2 \partial - p_F^{-2} \Delta) \right) \psi^-_{\mathbf{x}, t, \omega_2} \\ & + \int d\mathbf{x}\, d\mathbf{y}\, dt \int \prod_j d\omega_j \exp\{ip_F[(\omega_1 - \omega_4)\mathbf{x} + (\omega_2 - \omega_3)\mathbf{y}]\} \\ & \times \lambda_0(\mathbf{x} - \mathbf{y})\, \psi^+_{\mathbf{x}, t, \omega_1} \psi^+_{\mathbf{y}, t, \omega_2} \psi^-_{\mathbf{y}, t, \omega_3} \psi^-_{\mathbf{x}, t, \omega_4} \end{aligned} \tag{5.5}$$

We call (5.5) an *interaction* between the *Euclidean quasiparticles* with internal momentum ωp_F and position $\mathbf{x}$: it is obtained by considering the argument of the exponential in the rhs of (4.12) and by replacing the $\psi^{\pm}_{\mathbf{x}}$ fields via the (5.2).

One cannot say that such objects are real particles, as they only arise as an artificial device similar to the device of decomposing a field into scales. But we can pretend that they are real because we can infer from their properties those of the system, hence those of the *true* particles: the name is chosen because they seem to enjoy properties analogous, in some respects, to those of the quasiparticles used in the Landau theory of Fermi liquids.

Before giving the definition of Fermi surface in terms of the quasiparticles (see Definitions 1 and 2 below), we try to substantiate our interpretation of the above remarks on quasiparticles by showing that the mean field theory can be easily rephrased in our formalism, and very naturally so, leading to self-consistency equations of the BCS gap equation type.

We can think of the Euclidean quasiparticles as described by the formal fermion integration proportional to

$$P(d\psi) = \left\{ \exp - \int \varepsilon(k, \omega)\, \psi^+_{k, \omega} \psi^-_{k, \omega}\, d\omega\, dk \right\} d\psi^+\, d\psi^- \tag{5.6}$$

where an ultraviolet cutoff p_0 is introduced to avoid dealing with ultraviolet problems.

Using the formal expression (5.6), it is possible to build a *mean field* theory and to draw some analogy with the BCS model.

Let

$$\lambda(\boldsymbol{\omega};\boldsymbol{\omega}') \equiv [\hat{\lambda}(p_F(\boldsymbol{\omega}-\boldsymbol{\omega}')) - \hat{\lambda}(p_F(\boldsymbol{\omega}+\boldsymbol{\omega}')]/2$$

where $\hat{\lambda}$ is the Fourier transform of some rotational invariant effective potential λ (not to be considered as directly related to the initial potential), and define

$$V_{BCS} = -\int d\mathbf{x}\, dt \int d\boldsymbol{\omega}\, d\boldsymbol{\omega}'\, \lambda(\boldsymbol{\omega};\boldsymbol{\omega}')\, \Lambda^{-1} \psi^+_{\mathbf{x},t,\boldsymbol{\omega}} \psi^+_{\mathbf{x},t,-\boldsymbol{\omega}} \psi^-_{\mathbf{x},t,\boldsymbol{\omega}'} \psi^-_{\mathbf{x},t,-\boldsymbol{\omega}'} \quad (5.7)$$

and try to analyze $\int (\exp -V_{BCS})\, P(d\psi)$. Here Λ is a normalization constant formally equal to $\delta_{d-1}(\mathbf{0})$ if $\delta_{d-1} = \delta$ is as defined in (4.7): it disappears from the calculations eventually and one should think of it as defined in terms of some cutoff parameter.

The idea behind the model (5.7) is that it represents an effective potential describing the system, after integrating out the ultraviolet modes and on some very large scale $2^{-h}p_0^{-1}$: then $\delta_{d-1}(\mathbf{0})$ is $(2^{-h}p_0^{-1})^{d-1}$, so that $\Lambda^{-1} = 2^{h(d-1)} p_0^{d-1}$.

One tries to find the Schwinger functions for (5.7) by assuming that one can replace $\psi^-_{\mathbf{x},t,\boldsymbol{\omega}'} \psi^-_{\mathbf{x},t,-\boldsymbol{\omega}'}$ by its average in the distribution const $\cdot$ $(\exp -V_{BCS})\, P(d\psi)$, setting

$$\langle \psi^-_{\mathbf{x},t,\boldsymbol{\omega}'} \psi^-_{\mathbf{x},t,\boldsymbol{\omega}''} \rangle = \delta(\boldsymbol{\omega}'+\boldsymbol{\omega}'')\, V(\boldsymbol{\omega}') \quad (5.8)$$

where $V(\boldsymbol{\omega}) \equiv -V(-\boldsymbol{\omega})$ is considered as unknown and to be determined self-consistently by using

$$\delta(\boldsymbol{\omega}+\boldsymbol{\omega}')\, V(\boldsymbol{\omega}) = \frac{\int e^{-\bar{V}_{BCS}} P(d\psi)\, \psi^-_{\mathbf{x},t,\boldsymbol{\omega}} \psi^-_{\mathbf{x},t,\boldsymbol{\omega}'}}{\int e^{-\bar{V}_{BCS}} P(d\psi)} \quad (5.9)$$

with

$$\bar{V}_{BCS} = -\frac{1}{2} \int d\mathbf{x}\, dt \int d\boldsymbol{\omega}\, [w(\boldsymbol{\omega})\, \psi^+_{\mathbf{x},t,\boldsymbol{\omega}} \psi^+_{\mathbf{x},t,-\boldsymbol{\omega}} + w(\boldsymbol{\omega})\, \psi^-_{\mathbf{x},t,-\boldsymbol{\omega}} \psi^-_{\mathbf{x},t,\boldsymbol{\omega}}] \quad (5.10)$$

where, regarding $\lambda(\boldsymbol{\omega};\boldsymbol{\omega}')$ before (5.7) as defining a convolution operator K, the function $w(\boldsymbol{\omega}) \equiv -w(-\boldsymbol{\omega})$ is defined by

$$w(\boldsymbol{\omega}) = \int V(\boldsymbol{\omega}')\, \lambda(\boldsymbol{\omega};\boldsymbol{\omega}')\, d\boldsymbol{\omega}' \equiv (KV)(\boldsymbol{\omega}) \quad (5.11)$$

The best way to evaluate (5.9) is to make use of the formal functional integration representing $P(d\psi)$ [see (5.6)] and thus considering, for $(\mathbf{k})_1 > 0$,

$$-(w\psi^+_{k,\omega}\psi^+_{-k,-\omega} + w\psi^-_{-k,-\omega}\psi^-_{k,\omega}) + (\varepsilon_+\psi^+_{k,\omega}\psi^-_{k,\omega} + \varepsilon_-\psi^+_{-k,-\omega}\psi^-_{-k,-\omega}) \tag{5.12}$$

with $\varepsilon_+ \equiv \varepsilon(k, \omega)$, $\varepsilon_- = \varepsilon(-k, -\omega)$.

The (5.12) can be rewritten, dropping at places the ω dependence to shorten the notations, in terms of

$$\begin{aligned} \psi^-_{k,\omega} &= c(k)\,\varphi^-_{0,k,\omega} + s(k)\,\varphi^+_{1,k,\omega}, & \psi^+_{k,\omega} &= c(k)\,\varphi^+_{0,k,\omega} + s(k)\,\varphi^-_{1,k,\omega} \\ \psi^+_{-k,-\omega} &= -s(k)\,\varphi^-_{0,k,\omega} + c(k)\,\varphi^+_{1,k,\omega}, & \psi^-_{-k,-\omega} &= -s(k)\,\varphi^+_{0,k,\omega} + c(k)\,\varphi^-_{1,k,\omega} \end{aligned} \tag{5.13}$$

with $(\mathbf{k})_1 > 0$ and s, c are defined, if $\eta \equiv [\varepsilon(k, \omega) + \varepsilon(-k, -\omega)]/2$, by

$$t(k) = \frac{s(k)}{c(k)}, \qquad \frac{2t(k)}{1 - t(k)^2} = \frac{\omega}{\eta}, \qquad [c(k)^2 - s(k)^2]\,w - 2\eta c(k)\,s(k) = 0 \tag{5.14}$$

Note that $\varepsilon_+ = -ik_0 A(k, \omega) + A'(k, \omega)$ with $A = a(k) + \beta\omega\mathbf{k}d(k)$ and $A' = b(k) + \beta\omega\mathbf{k}c(k)$.

We find that (5.12) becomes, for $(\mathbf{k})_1 > 0$,

$$\varepsilon_0(k, \omega)\,\varphi^+_0\varphi^-_0 + \varepsilon_1(k, \omega)\,\varphi^+_1\varphi^-_1 \qquad \begin{cases} \varepsilon_0 = c^2\varepsilon(k, \omega) - s^2\varepsilon(-k, -\omega) + 2wcs \\ \varepsilon_1 = c^2\varepsilon(-k, -\omega) - s^2\varepsilon(k, \omega) + 2wcs \end{cases} \tag{5.15}$$

i.e., $\varepsilon_0 = -ik_0 A(k, \omega) + B(k, \omega)$, $\varepsilon_1 = ik_0 A(k, \omega) + B(k, \omega)$, with $B(k, \omega) = (c^2 - s^2)\eta + 2wcs$, and we can easily compute

$$\begin{aligned} AV(\omega) &= \langle \psi^-_{\mathbf{x},t,\omega}\psi^-_{\mathbf{x},t,-\omega}\rangle \\ &= \int \frac{dk}{(2\pi)^{d+1}}\,[s(k, \omega)\,c(k, \omega)(\langle \varphi^+_{0,k,\omega}\varphi^-_{0,k,\omega}\rangle + \langle \varphi^+_{1,k,\omega}\varphi^-_{1,k,\omega}\rangle)] \\ &\equiv -A\int \frac{dk}{(2\pi)^{d+1}}\,s(k, \omega)\,c(k, \omega)\left(\frac{1}{\varepsilon_0(k, \omega)} + \frac{1}{\varepsilon_1(k, \omega)}\right) \end{aligned} \tag{5.16}$$

One can see from (5.14) and (5.15) that the sign of Bsc is the same as that of w, so that (5.16) becomes

$$V(\omega) = -\int \frac{dk}{(2\pi)^{d+1}}\,\frac{w(\omega)}{[\eta^2 + w(\omega)^2]^{1/2}}\,\frac{|B(k)|}{k_0^2 A(k)^2 + B(k)^2} \tag{5.17}$$

In (5.17) the large values of k do not matter too much because the functions A, B, η diverge fast enough for summability. Therefore for $d=1$, using $A(0)=1+O(k^2)$, $\eta(k)=\beta\omega\mathbf{k}+O(k^2)$ and assuming for the purpose of an example $\lambda(\omega,\omega')=\omega\omega' V_0$ and if $V(\omega)=\omega V$, $w(\omega)=V_0 V\omega$, the self-consistency relation (5.17) for V becomes

$$\frac{1}{V_0} \simeq -\int \frac{d\mathbf{k}}{4\pi} \frac{1}{(\eta^2+V^2V_0^2)^{1/2}}$$

$$\simeq -\int \frac{d\mathbf{k}}{4\pi} \frac{1}{[(\mathbf{k}^2-p_F^2)^2/2m+V_0^2V^2]^{1/2}} \tag{5.18}$$

and the rhs has been obtained by changing variables from $\mathbf{k}$ to $\mathbf{k}+p_F\boldsymbol{\omega}$. Equation (5.18) is formally the well-known *BCS gap equation*[14] (usually written calling $V^2V_0^2\equiv\Delta$). Equation (5.18) is a self-consistency equation which can be solved under the hypothesis that the potential is attractive [i.e., $\lambda(\mathbf{x}-\mathbf{y})\leqslant 0$, implying that $V_0<0$] and not soluble if it is repulsive [i.e., $\lambda(\mathbf{x}-\mathbf{y})\geqslant 0$].

For $d>1$ the situation is very similar and one expects that the self-consistency equation will be soluble if the potential is attractive: rather than performing a general analysis of (5.17), we show the truth of the above statements in some special cases. The basic remark is that $\int (dk/|\eta|)(|B|/k_0^2A^2+B^2)$ diverges near $k=0$.

Consider first the *δ-model*:

$$\lambda(\boldsymbol{\omega};\boldsymbol{\omega}')=V_0\,\tfrac{1}{2}[\delta(\boldsymbol{\omega}-\boldsymbol{\omega}')-\delta(\boldsymbol{\omega}+\boldsymbol{\omega}')] \tag{5.19}$$

corresponding to a potential with Fourier transform such that $\hat{\lambda}(p_F(\boldsymbol{\omega}-\boldsymbol{\omega}'))\equiv V_0\delta(\boldsymbol{\omega}-\boldsymbol{\omega}')$. This is a singular but interesting case; the self-consistent V is easy to calculate explicitly and, if $\boldsymbol{\omega}_z$ denotes the z component of $\boldsymbol{\omega}$ with respect to a pre-fixed z axis, it is $V(\boldsymbol{\omega})=[\mathrm{sign}(\boldsymbol{\omega}_z)]V$, with V being a constant defined by the equation

$$\frac{1}{V_0}=-\int \frac{d^4k}{(2\pi)^4} \frac{1}{[\eta^2+(VV_0)^2]^{1/2}} \frac{|B(k)|}{k_0^2A(k)^2+B(k)^2} \tag{5.20}$$

again very similar to the BCS gap equation, and which has nontrivial solutions in the case $V_0<0$, i.e., if the interaction is attractive.

A second interesting case is the *P-wave model* in which the Fourier transform of the potential is $\hat{\lambda}(p_F(\boldsymbol{\omega}-\boldsymbol{\omega}'))=-3(\boldsymbol{\omega}-\boldsymbol{\omega}')^2\,V_1/2$, so that $\lambda(\boldsymbol{\omega};\boldsymbol{\omega}')=3\boldsymbol{\omega}\boldsymbol{\omega}'V_1$; in this case we find that $w(\boldsymbol{\omega})$ is a P-wave function $w(\boldsymbol{\omega})=3\boldsymbol{\omega}_z VV_1$, where V is the (unknown) coefficient of the P_1 Legendre polynomial in the Legendre expansion of $V(\boldsymbol{\omega})$:

$$\frac{1}{V_1}=-\int \frac{\sin\vartheta\,d\vartheta}{2} \int \frac{d^4k}{(2\pi)^4} \frac{3\cos^2\vartheta}{(\eta^2+V^2V_1^2\cos^2\vartheta)^{1/2}} \frac{|B(k)|}{A(k)^2+B(k)^2} \tag{5.21}$$

which again admits a solution only if $V_1 < 0$, i.e., only if the potential is attractive. It is the extension to P wave, in our spinless mode, of the gap equation. Once V_1 is determined, the other coefficients of the Legendre expansion of V are trivially determined by (5.17), whose rhs becomes a wholly known quadrature.

More generally, one should remark that the function $\lambda(\boldsymbol{\omega}; \boldsymbol{\omega}')$ can be expanded in Legendre series:

$$\lambda(\boldsymbol{\omega}; \boldsymbol{\omega}') = \sum_{l=\text{odd}} (2l+1)\, \lambda_l P_l(\boldsymbol{\omega} \cdot \boldsymbol{\omega}') \tag{5.22}$$

and one easily checks that the condition $\lambda(\mathbf{x}-\mathbf{y}) \geqslant 0$ implies that $\lambda_l \geqslant 0$. In fact one regards the $\lambda(\boldsymbol{\omega}; \boldsymbol{\omega}')$ as a convolution kernel K on the odd functions on the sphere and checks that its quadratic form can be written as

$$(f, Kf) = \int \lambda(\mathbf{x})\, |F_f(\mathbf{x})|^2\, d^3\mathbf{x} \geqslant 0 \qquad \text{if} \quad F_f(\mathbf{x}) = \int e^{ip_F \boldsymbol{\omega}\mathbf{x}} f(\boldsymbol{\omega})\, d\boldsymbol{\omega} \tag{5.23}$$

on the other hand, λ_l are just the eigenvalues of K as a convolution kernel (which has the spherical harmonics as eigenfunctions).

The gap equation (5.17) for a model with *finitely many waves*

$$\lambda(\boldsymbol{\omega}; \boldsymbol{\omega}') = \sum_{l=\text{odd}}^{2N+1} (2l+1)\, \lambda_1 P_l(\boldsymbol{\omega} \cdot \boldsymbol{\omega}') \tag{5.24}$$

becomes, as it is easily checked, a system of N equations and N unknowns which determine the first N Legendre series coefficients of $V(\boldsymbol{\omega})$; the other coefficients (infinitely many) are trivially determined in terms of the first N. The reason for this simplification is that $w = KV$, and K in this case has N-dimensional range spanned by the first N odd Legendre polynomials: hence the rhs of (5.17) is wholly known once the first N components of V are known (because $KV = w$ depends only on them).

We should therefore expect that perturbation theory can be consistent only if the interaction is repulsive and this will be an important guideline in the coming analysis. Furthermore, in some sense only the components of the quasiparticle potential corresponding to waves that appear in the initial interaction should be *relevant*: they are the ones that determine the self-consistent solution.

The above analysis shows, in our opinion, the interest of the new notion of quasiparticle. Hence we are motivated to provide a definition of the Fermi surface in terms of quasiparticles.

We simply look at the truncated Schwinger functions $S_{(\geqslant h)}$ considered at the end of Section 4, with ultraviolet cutoff at p_0 and infratred cutoff at $2^h p_0$, for the ordinary fields and defined with the obvious changes for quasiparticle fields.

In the free case they can be easily computed from (5.1) and from the analysis of Appendix A.

We investigate the Schwinger function $S_{(\geqslant h)}(x-y)$ in the presence of interaction in terms of the free and interacting quasiparticle Schwinger functions $g_{(\geqslant h)}(x-y, \boldsymbol{\omega})\,\delta(\boldsymbol{\omega}-\boldsymbol{\omega}')$ and $S_{(\geqslant h)}(x, \boldsymbol{\omega}, y, \boldsymbol{\omega}')$. In this case the function $S_{(\geqslant h)}$ is defined in terms of the effective potential via the obvious generalization of formula (3.9):

$$\begin{aligned} &S_{(\geqslant h)}(x, \boldsymbol{\omega}, y, \boldsymbol{\omega}') \\ &\quad = g_{(\geqslant h)}(x-y, \boldsymbol{\omega})\,\delta(\boldsymbol{\omega}-\boldsymbol{\omega}') - \int dz\, dz' e^{ip_F(\boldsymbol{\omega}\mathbf{z}-\boldsymbol{\omega}'\mathbf{z})} \\ &\quad \times g_{(\geqslant h)}(x-z, \boldsymbol{\omega})\, V^{(h)}_{\text{eff}}(z, \boldsymbol{\omega}, \mathbf{z}', \boldsymbol{\omega}')\, g_{(\geqslant h)}(z'-y, \boldsymbol{\omega}') \end{aligned} \tag{5.25}$$

where the coefficient $\exp ip_F(\boldsymbol{\omega}\mathbf{z}-\boldsymbol{\omega}'\mathbf{z}')$ is extracted for convenience, as suggested by (5.3), from the kernels defining the effective potential, which we introduce via the relations

$$\begin{aligned} e^{-V^{(h)}_{\text{eff}}(\psi)} &= \int P(d\psi^{(h+1)}\, d\psi^{(h+2)} \cdots) \\ &\quad \times \exp[-V^{(0)}(\psi + \psi^{(h+1)} + \psi^{(h+2)} \cdots)] \end{aligned} \tag{5.26}$$

We now suppose that

$$\begin{aligned} V^{(h)}_{\text{eff}}(z, \boldsymbol{\omega}, z', \boldsymbol{\omega}') &= 2^h v_h \delta(z-z') + \alpha_h \delta(z-z')(i\beta\boldsymbol{\omega}'\partial' - \Delta'/2m) \\ &\quad + \zeta_h \delta(z-z')\partial'_t + w'_h(z, z') \end{aligned} \tag{5.27}$$

where ∂'_t, ∂', and Δ' are differential operators acting to their right on the z' variables; and we see that the reconstruction of the Schwinger function

$$S_{(\geqslant h)}(x, y) = \int \{\exp[-ip_F(\boldsymbol{\omega}\mathbf{x}-\boldsymbol{\omega}'\mathbf{y})]\}\, S_{(\geqslant h)}(x, \boldsymbol{\omega}, y, \boldsymbol{\omega}')\, d\boldsymbol{\omega}\, d\boldsymbol{\omega}' \tag{5.28}$$

leads to

$$\begin{aligned} &S_{(\geqslant h)}(x-y) \\ &\quad = g_{(\geqslant h)}(x-y) - \int d\boldsymbol{\omega}'\, dz'\; g_{(\geqslant h)}(x-z') \exp[-ip_F\boldsymbol{\omega}'(\mathbf{z}'-\mathbf{y})] \\ &\qquad \times [2^h v_h + \alpha_h(i\beta\boldsymbol{\omega}'\partial - \Delta/2m) + \zeta_h\partial_t]\; g_{(\geqslant h)}(z'-y, \boldsymbol{\omega}') + \cdots \\ &\quad = g_{(\geqslant h)}(x-y) \\ &\qquad - \int g_{(\geqslant h)}(x-z')[2^h v_h + \zeta_h\partial_t + \alpha_h(-\Delta - p_F^2)/2m]\; g_{(\geqslant h)}(z'-y)\, dz' \end{aligned} \tag{5.29}$$

and in Fourier transform

$$\hat S_{(\geq h)}(k) = \hat g_{(\geq h)}(k) - \hat g_{(\geq h)}(k)\, \hat g_{(\geq h)}(k) \times [2^h \nu_h + \zeta_h(-ik_0) + \alpha_h(\mathbf{k}^2 - p_F^2/2m) + \cdots] \tag{5.30}$$

But by our construction $g_{(\geq h)}(k)$ has a singularity, at the Fermi surface when $h \to -\infty$, i.e., for $k_0 = 0$, $|\mathbf{k}| = p_F$:

$$\frac{1 - \exp\{-2^{-2h}[k_0^2 + (\mathbf{k}^2 - p_F^2/2m)^2]\}}{-ik_0 + (\mathbf{k}^2 - p_F^2/2m)} \tag{5.31}$$

Hence we see that the singularity at the Fermi surface, i.e., at $k_0 = 0$, $|\mathbf{k}| = p_F$, is not changed if

$$\nu_{-\infty} = 0, \qquad \zeta_{-\infty} = \alpha_{-\infty} \tag{5.32}$$

provided $\nu_h \to 0$ at least as fast as $2^{\varepsilon h}$, $\varepsilon > 0$.

This can be interpreted as saying that the $S_{(\geq h)}$ function has the same behavior at ∞ as the corresponding function in the free case, provided the remainder gives a less singular contribution; this happens if the contribution of the remainder $w_h(z - z')$ to the effective potential is supposed to have a Fourier transform $\hat w'_h(k)$ uniformly well behaved near $k_0 = 0$, $|\mathbf{k}| = p_F$ and vanishing there to first order:

$$\int dz\, e^{ik(z-z')} w'_h(z - z')\, dz' = \hat w'_h(k) \equiv \hat w'_h(k_0, \mathbf{k})$$

$$\hat w'_h(p_0, p_F\boldsymbol{\omega}) = 0, \qquad \frac{\partial}{\partial p} \hat w'_h(p_0, p\boldsymbol{\omega})\Big|_{\substack{p = p_F \\ p_0 = 0}} = 0, \qquad \frac{\partial}{\partial p_0} \hat w'_h(p_0, p\boldsymbol{\omega})\Big|_{\substack{p = p_F \\ p_0 = 0}} = 0 \tag{5.33}$$

Our definition of existence of a normal Fermi surface and of the mass of the quasiparticles is based on the above heuristic arguments:

Definition 1. We say that there is a normal Fermi surface of radius p_F and there are quasiparticles of mass m if the part of the effective potential on scale h and degree 2 in the fields (i.e., the part connected with the interaction propagator) satisfies

1. $\nu_h \to 0$; α_h, $\zeta_h \to \zeta_\infty$.
2. $w'_h(z - z')$ is a distribution with Fourier transform uniformly regular within a momentum layer of size $O(2^{(1-\varepsilon)h} p_0)$, for some $\varepsilon > 0$, near $k = (0, \mathbf{k})$ with $|\mathbf{k}| = p_F$, and satisfying (5.33).

In fact, we think of property 1 as the weakest definition of a normal Fermi surface and quasiparticle mass, and we think of the pair 1, 2 as a stronger version: we refer to the two possible meanings by saying, respectively, *in the sense of definition* (1, 1) or (1, 2).

Note that it would be perhaps more satisfactory to require property 2 for a layer around the Fermi surface of width $O(p_0)$ rather than $O(2^{(1-\varepsilon)h}p_0)$: however, we are unable to study this stronger property, and the reason is quite fundamental (Section 11). The physical meaning is that the pair Schwinger function behaves as the free one (with the corresponding Fermi surface behavior) only if we look at distances which grow very large with the infrared cutoff. We must therefore couple the thermodynamic limit with the computation of the correlation function: for fixed $\varepsilon > 0$ (*a priori*), the correlation has to be computed at distances between $L^{1-\varepsilon}$ and L if the system is confined in a box of size L and if we want to see the Fermi surface singularity with no corrections.

Setting $\Delta^2 = k_0^2 + (\mathbf{k}^2 - p_F^2)$, we develop, in Section 12, an expansion for the Schwinger functions implying that the corresponding expansion for the function $w_h'(k)/\Delta^{1+\varepsilon}$ is bounded uniformly if $p_0 2^h < \Delta < p_0 2^{h(1-\varepsilon)}$ for $\varepsilon < 1/4$.

We also show that *all* the effective potentials (connected with the many-body Schwinger functions) admit an expansion in powers of α_h, ζ_h, ν_h and of a fourth *running form factor* λ_h, which is a function on $(S_F)^4$, S_F being the Fermi sphere. The expansion has coefficients bounded, to order n, by $n!$, using the supremum norm to measure the size (see Section 9) of λ_h.

Furthermore, the $\mathbf{v}_h = (\nu_h, \alpha_h, \zeta_h, \lambda_h)$ are given by a formal power series in $\mathbf{v}_{h+1}$, $\mathbf{v}_{h+2}$,... with coefficients bounded by $n!$:

$$\mathbf{v}_{h-1} = A\mathbf{v}_h + B(\mathbf{v}_h, \mathbf{v}_{h+1}, \ldots) \tag{5.34}$$

with A a diagonal matrix with diagonal (2, 1, 1, 1) and B, called the *beta functional*, given by a formal power series in its arguments $\mathbf{v}$.

Hence we have a way to define what we mean by a *perturbatively well-defined* normal Fermi surface and quasiparticles.

Definition 2. Consider the relation obtained from (5.34) by truncating B to a given order p in its expansion in the $\mathbf{v}$'s. If the new truncated relation generates, for suitably chosen α_0, ν_0 and for given $\zeta_0 = 0$ and λ_0 small enough, a sequence $\mathbf{v}_h$ bounded uniformly and such that $\nu_h \to 0$ and $\alpha_h - \zeta_h \to 0$, then we say that perturbation theory is *consistent up to order* p and that, to this order, the normal Fermi surface exists together with quasiparticles of mass m [in the sense of definition (1, 1)].

In Sections 6–13 we develop a general perturbation theory aimed at studying normal Fermi surfaces.

But there is also a natural notion of *anomalous Fermi surface*: we shall see in Section 15 that one-dimensional spinless systems show anomalous Fermi surfaces and that our formalism can be easily adapted to the theory of such anomalous Fermi surfaces.

Without entering into a heuristic analysis for the purpose of motivation, one can say that a system has an anomalous Fermi surface when the numbers α_h, ζ_h approach a singular value (i.e., ∞) in the limit $h \to -\infty$. Setting $Z_h = 1 + \zeta_h$, we say that the anomalous dimension of the Fermi surface is η if $Z_h \asymp 2^{-\eta h}$; see Section 15 for a more precise discussion.

Our analysis does not extend to the theory of anomalous Fermi surfaces in $d > 1$ systems. However, it seems likely that systems with short-range repulsive forces show normal Fermi surfaces if $d > 1$; see Section 14. At the same time, when $d > 1$, new interesting phenomena become possible, such as the concentration of the interaction in the *Cooper pairs* of quasiparticles and the relevance of the interaction sign.

6. RELEVANT VERSUS IRRELEVANT

The flow of the effective potential will be analyzed by using the methods of the renormalization group.[16–18] We begin with

$$V^{(0)} = V(\psi^{(\leqslant 0)}) + \nu_0 N(\psi^{(\leqslant 0)}) + \alpha_0 T(\psi^{(\leqslant 0)}) + \text{const} \tag{6.1}$$

and as a first step, to adhere as much as possible to the formalism of refs. 16 and 17, we rewrite (6.1) in terms of Wick ordered expressions in the quasi particle fields, and of the *covariant derivative operator* $\mathscr{D}^{\pm} = (\partial_t, \partial_{\mathbf{x}} \mp i(\boldsymbol{\omega}/2p_{\mathrm{F}})\Delta) = (\mathscr{D}_t, \mathscr{D}^{\pm}_{\boldsymbol{\omega}})$:

$$\begin{aligned} V^{(0)} = & \int d\mathbf{x}\, dt \prod_{i=1}^{2} d\boldsymbol{\omega}_i\, (\nu_0 + \tilde{\nu}_0)\, e^{ip_{\mathrm{F}}(\omega_1 - \omega_2)\mathbf{x}} :\psi^{+}_{\mathbf{x},t,\omega_1} \psi^{-}_{\mathbf{x},t,\omega_2}: \\ & + \int d\mathbf{x}\, d\mathbf{y}\, dt \prod_{i=1}^{2} d\boldsymbol{\omega}_i\, \tilde{\nu}_1(\mathbf{x}-\mathbf{y})\, e^{ip_{\mathrm{F}}(\omega_1 \mathbf{x} - \omega_2 \mathbf{y})} :\psi^{+}_{\mathbf{x},t,\omega_1} \psi^{-}_{\mathbf{y},t,\omega_2}: \\ & + \int d\mathbf{x}\, dt\, \alpha_0 \beta \prod_{i=1}^{2} d\boldsymbol{\omega}_i\, e^{ip_{\mathrm{F}}(\omega_1 - \omega_2)\mathbf{x}} :\psi^{+}_{\mathbf{x},t,\omega_1} i\omega_2 \mathscr{D}_{\omega_2} \psi^{-}_{\mathbf{x},t,\omega_2}: \\ & + \int d\mathbf{x}\, d\mathbf{y}\, dt \prod_{i=1}^{4} d\boldsymbol{\omega}_i\, e^{ip_{\mathrm{F}}[(\omega_1 - \omega_4)\mathbf{x} - (\omega_3 - \omega_2)\mathbf{y}]} \lambda_0(\mathbf{x}-\mathbf{y}) \\ & \times :\psi^{+}_{\mathbf{x},t,\omega_1} \psi^{+}_{\mathbf{y},t,\omega_2} \psi^{-}_{\mathbf{y},t,\omega_3} \psi^{-}_{\mathbf{x},t,\omega_4}: \end{aligned} \tag{6.2}$$

where we write $\mathscr{D}$ for $\mathscr{D}^-$, and $g(\mathbf{x}, t, \boldsymbol{\omega}) = \sum_{-\infty}^{0} g_n(\mathbf{x}, t, \boldsymbol{\omega})$ is defined by (4.5); rotational invariance is used, and we set

$$\tilde{v}_0 = -2\hat{\lambda}_0(\mathbf{0}) \int g(\mathbf{0}, 0, \boldsymbol{\omega})\, d\boldsymbol{\omega}, \qquad \hat{\lambda}_0(\mathbf{k}) = \int d\mathbf{x}\, \lambda_0(\mathbf{x})\, e^{-i\mathbf{k}\mathbf{x}}$$
$$\tilde{v}_1(\mathbf{z}) = 2 \int g(\mathbf{z}, 0, \boldsymbol{\omega})\, \lambda_0(\mathbf{z})\, e^{-ip_F\boldsymbol{\omega}\mathbf{z}}\, d\boldsymbol{\omega} \tag{6.3}$$

Here, as well as in the following sections, $\boldsymbol{\omega}_2$ will often be regarded as a four-vector with vanishing t component.

The use of Wick ordering could in fact be easily avoided; see concluding remarks of Section 10.

While (6.2) looks quite complicated, it can be split into two parts which will be called, respectively, *relevant* and *irrelevant*.

The relevant part of $V^{(0)}$ consists of the *local part* $V_L^{(0)}$ of $V^{(0)}$ defined by

$$\begin{aligned} V_L^{(0)} = & \int \prod_{i=1}^{2} d\boldsymbol{\omega}_i\, d\mathbf{x}\, dt\, (v_0 + \tilde{v}_0)\, e^{ip_F(\boldsymbol{\omega}_1 - \boldsymbol{\omega}_2)\mathbf{x}} :\psi^+_{\mathbf{x},t,\boldsymbol{\omega}_1} \psi^-_{\mathbf{x},t,\boldsymbol{\omega}_2}: \\ & + \int \prod_{i=1}^{2} d\boldsymbol{\omega}_i\, d\mathbf{x}\, d\mathbf{y}\, dt\, \tilde{v}_1(\mathbf{x}-\mathbf{y})\, e^{ip_F(\boldsymbol{\omega}_1\mathbf{x} - \boldsymbol{\omega}_2\mathbf{y})} \\ & \times :\psi^+_{\mathbf{x},t,\boldsymbol{\omega}_1}(\psi^-_{\mathbf{x},t,\boldsymbol{\omega}_2} + (\mathbf{y}-\mathbf{x})\, \mathscr{D}_{\boldsymbol{\omega}_2} \psi^-_{\mathbf{x},t,\boldsymbol{\omega}_2}): \\ & + \alpha_0 \beta \int \prod_{i=1}^{2} d\boldsymbol{\omega}_i\, d\mathbf{x}\, dt\, e^{ip_F(\boldsymbol{\omega}_1 - \boldsymbol{\omega}_2)\mathbf{x}} :\psi^+_{\mathbf{x},t,\boldsymbol{\omega}_1} i\boldsymbol{\omega}_2 \mathscr{D}_{\boldsymbol{\omega}_2} \psi^-_{\mathbf{x},t,\boldsymbol{\omega}_2}: \\ & - \int d\mathbf{x}\, dt \prod_{i=1}^{4} d\boldsymbol{\omega}_i\, e^{ip_F(\boldsymbol{\omega}_1 + \boldsymbol{\omega}_2 - \boldsymbol{\omega}_3 - \boldsymbol{\omega}_4)\mathbf{x}} \lambda_0(\boldsymbol{\omega}_1, \boldsymbol{\omega}_2, \boldsymbol{\omega}_3, \boldsymbol{\omega}_4) \\ & \times :\psi^+_{\mathbf{x},t,\boldsymbol{\omega}_1} \psi^+_{\mathbf{x},t,\boldsymbol{\omega}_2} \psi^-_{\mathbf{x},t,\boldsymbol{\omega}_3} \psi^-_{\mathbf{x},t,\boldsymbol{\omega}_4}: \end{aligned} \tag{6.4}$$

where we have set, or we set for later use, $\hat{\lambda}_0(\boldsymbol{\omega} - \boldsymbol{\omega}') \equiv \hat{\lambda}_0(p_F(\boldsymbol{\omega} - \boldsymbol{\omega}'))$ and

$$\begin{aligned} \bar{v}_0 &= v_0 + \tilde{v}_0 + \int \tilde{v}_1(\mathbf{z})\, e^{ip_F\boldsymbol{\omega}\mathbf{z}}\, d\mathbf{z} \\ \bar{\alpha}_0 &= \alpha_0 + i\beta^{-1}\boldsymbol{\omega} \int e^{ip_F\boldsymbol{\omega}\mathbf{z}} \tilde{v}_1(\mathbf{z})\, \mathbf{z}\, d\mathbf{z} \\ \lambda_0(\omega_1, \omega_2, \omega_3, \omega_4) &= -\tfrac{1}{4}[\hat{\lambda}_0(\boldsymbol{\omega}_2 - \boldsymbol{\omega}_3) - \hat{\lambda}_0(\boldsymbol{\omega}_1 - \boldsymbol{\omega}_3) \\ &\quad - \hat{\lambda}_0(\boldsymbol{\omega}_2 - \boldsymbol{\omega}_4) + \hat{\lambda}_0(\boldsymbol{\omega}_1 - \boldsymbol{\omega}_4)] \end{aligned} \tag{6.5}$$

Hence the *relevant part* of the interaction can be written as

$$V_L^{(0)} = \bar{\nu}_0 \int \prod_{i=1}^{2} d\omega_i \, d\mathbf{x} \, dt \, e^{ip_F(\omega_1 - \omega_2)\mathbf{x}} :\psi^+_{\mathbf{x},t,\omega_1}\psi^-_{\mathbf{x},t,\omega_2}:$$
$$+ \bar{\alpha}_0 \int \prod_{i=1}^{2} d\omega_i \, d\mathbf{x} \, dt \, e^{ip_F(\omega_1 - \omega_2)\mathbf{x}} :\psi^+_{\mathbf{x},t,\omega_1}(i\beta\omega_2 \mathscr{D}_{\omega_2}\psi^-_{\mathbf{x},t,\omega_2}):$$
$$- \int d\mathbf{x} \, dt \prod_{i=1}^{4} d\omega_i \, e^{ip_F(\omega_1+\omega_2-\omega_3-\omega_4)\mathbf{x}} \lambda_0(\omega_1, \omega_2, \omega_3, \omega_4)$$
$$\times :\psi^+_{\mathbf{x},t,\omega_1}\psi^+_{\mathbf{x},t,\omega_2}\psi^-_{\mathbf{x},t,\omega_3}\psi^-_{\mathbf{x},t,\omega_4}: \tag{6.6}$$

So the initial potential $V^{(0)}$ is split as

$$V^{(0)} = V_L^{(0)} + V_R^{(0)} \tag{6.7}$$

and the reason why we call the first *relevant* and the second *irrelevant* will become clear later and is at least twofold:

1. We shall see that if we set $V_R^{(0)} \equiv 0$, we find no more and no fewer difficulties in developing the theory of the flow of $V^{(n)}$, $n = -1, -2, \ldots$.

2. The $V_L^{(0)}$ will turn out to be composed of three *marginal* terms [the λ_0 term, the $\bar{\alpha}_0$ term, and the ζ_0 term, see (4.12), which is initially zero] and one *relevant* term (the $\bar{\nu}_0$ term), while the $V_R^{(0)}$ will only contain *irrelevant terms*, the above words being used in the sense of the renormalization group theory.[16–18]

7. LOCALIZATION OPERATORS

We follow the methods developed in refs. 16–18 to analyze the flow of $V^{(h)}$ as $h \to -\infty$. Therefore we must introduce localization operators which, acting on a potential $V^{(h)}$ expressed as a sum of Wick monomials, turn it into an expression like (6.6).

We shall operate on V's which are written as sums of integrals of Wick monomials P in the fields, multiplied by regular (i.e., nondistributions) kernels W generated by well-defined rules (*Feynman diagrams*). The kernels depend on the field labels and the first space-time label x of the first field in P will be called the *localization point* of the monomial P. We adopt systematically the convention that $x \equiv (\mathbf{x}, t)$.

Each of such integrals is an expression O_P which will be called an *operator* contributing to $V^{(h)}$. If the Wick monomial depends only on fields computed at the same point, then we say that the operator is *local*. The kernel W is called a *form factor* for O_P.

The most convenient way to define a localization operator $\mathscr{L}_h$ acting on such V's is to describe how it acts on the elementary monomials P and extend it by linearity.

The action of $\mathscr{L}_h$ on the operators will be analogous to the operation described in the previous section in going from $V^{(0)}$ to $V_L^{(0)}$, which is a *localization* in a strict sense, i.e., it turns some nonlocal operators into local operators.

The Wick monomials that we shall consider will be monomials in the *local* fields $\psi_{x\omega}^{(\leqslant h)\pm}$,

$$\mathscr{D}\psi_{x\omega}^{(\leqslant h)-} \equiv [\partial_t, \partial_{\mathbf{x}} + i(\omega/2p_F)\Delta]\, \psi_{x\omega}^{(\leqslant h)-} \equiv (\mathscr{D}_t, \mathscr{D}_\omega)\, \psi_{x\omega}^{(\leqslant h)-}$$

and in the *nonlocal* fields:

$$\begin{aligned} D_{x'x\omega}^{(\leqslant h)\pm} &= (\psi_{x'\omega}^{(\leqslant h)\pm} - \psi_{x\omega}^{(\leqslant h)\pm}) \\ S_{x'x\omega}^{1(\leqslant h)} &= (\psi_{x'\omega}^{(\leqslant h)-} - \psi_{x\omega}^{(\leqslant h)-} - (x'-x)\, \mathscr{D}\psi_{x\omega}^{(\leqslant h)-}) \\ S_{x'x\omega}^{2(\leqslant h)} &= \mathscr{D}\psi_{x'\omega}^{(\leqslant h)-} - \mathscr{D}\psi_{x\omega}^{(\leqslant h)-} \\ S_{x_1x_2x_3x_4\omega}^{3(\leqslant h)} &= (x_3 - x_4)\, S_{x_1x_2\omega}^{2} \end{aligned} \tag{7.1}$$

where we call *sites* of the nonlocal fields the set of space-time indices appearing in them. We can similarly define the nonlocal field components $D_{xy\omega}^{(h)}$, $S_{xy\omega}^{1(h)}$, etc., using $\psi^{(h)}$, $\mathscr{D}\psi^{(h)}$ instead of $\psi^{(\leqslant h)}$, $\mathscr{D}\psi^{(\leqslant h)}$.

The operator $\mathscr{D} \equiv \mathscr{D}^-$ is the covariant derivative operator introduced in the previous section; see (6.2). The fields D, S^i are thought of as emerging from the point appearing as their first space-time label.

If $A^{(h)}$, $B^{(h)}$ are two of the above fields and d_A denotes the vector joining the first two sites of A or the tensor formed by the two vectors connecting the first site to the second site and the third to the fourth (if there is only one site, $d_A \equiv 1$), we see from (4.5) and (7.1) that the free Schwinger functions $\langle A^{(h)} B^{(h)} \rangle$ have the form

$$\langle A^{(h)} B^{(h)} \rangle = 2^{(\tau_A + 1/2)h} 2^{(\tau_B + 1/2)h} (2^h d_A)^{\gamma_A} (2^h d_B)^{\gamma_B} g_h^{AB} \tag{7.2}$$

where $\tau_A = 0$ if A is ψ, D, S^1, S^3, $\tau_A = 1$ if A is $\mathscr{D}\psi$, S^2; and γ_A is the formal order of zero of A, i.e., the homogeneity degree of d_A (here $d_A^{\gamma_A}$ is a notation for a tensor with γ_A indices); g_h^{AB} is a function (with some small dependence on h) of the set of sites of A and B scaled by 2^h. Given any Wick monomial P in the above fields A, we define

$$\begin{aligned} z_P &= \sum_{A \in P} \gamma_A = \{\text{dimensional gain of } P\}, \qquad d_P = \prod_{A \in P} d_A^{\gamma_A} \\ \delta_P^0 &= -2 + \sum_{A \in P} (\tfrac{1}{2} + \tau_A) = \{\text{scaling dimension of } O_P\} \\ \delta_P &= -2 + \sum_{A \in P} (\tfrac{1}{2} + \tau_A + \gamma_A) = \{\text{dimension of } O_P\} \end{aligned} \tag{7.3}$$

with d_P a tensor with z_P indices.

The above *dimension-independent* power counting arises from the fact that the particle propagator singularity is on a surface of codimension 2 (namely, the Fermi surface $k_0 = 0$, $|\mathbf{k}| = p_F$), independent of d; see also the comments after (10.9) below.

Thus we see that the operators in the relevant part (6.6) of the interaction give rise to operators O with nonpositive dimension, while the (nonlocal) terms of the irrelevant part of the effective potential are operators with positive dimension.

In fact, we can write the irrelevant part [defined as the difference $V_R^{(0)}$ between $V^{(0)}$ in (6.2) and $V_L^{(0)}$ in (6.6)] by comparing (6.6) to (6.2), via (6.5) and (6.3), and getting

$$\begin{aligned} V_R^{(0)} = & \int d\mathbf{x}\, d\mathbf{y}\, dt\, d\boldsymbol{\omega}_1\, d\boldsymbol{\omega}_2\, e^{ip_F(\omega_1-\omega_2)x}\tilde{v}_1(\mathbf{x}-\mathbf{y}):\psi^+_{x\omega_1} S^1_{yx\omega_2}: \\ & + \int d\mathbf{x}\, d\mathbf{y}\, dt\, d\boldsymbol{\omega}_1\, d\boldsymbol{\omega}_2\, d\boldsymbol{\omega}_3\, d\boldsymbol{\omega}_4\, e^{ip_F[(\omega_1-\omega_4)x+(\omega_2-\omega_3)y]}\lambda(\mathbf{y}-\mathbf{x}) \\ & \times 2^{-1}(:\psi^+_{\mathbf{x}t\omega_1} D^+_{\mathbf{y}t\mathbf{x}t\omega_2}\psi^-_{\mathbf{y}t\omega_3}\psi^-_{\mathbf{x}t\omega_4}: + :\psi^+_{\mathbf{x}t\omega_1}\psi^+_{\mathbf{x}t\omega_2} D^-_{\mathbf{y}t\mathbf{x}t\omega_3}\psi^-_{\mathbf{x}t\omega_4}: \\ & + :D^+_{\mathbf{x}t\mathbf{y}t\omega_1}\psi^+_{\mathbf{y}t\omega_2}\psi^-_{\mathbf{y}t\omega_3}\psi^-_{\mathbf{x}t\omega_4}: + :\psi^+_{\mathbf{y}t\omega_1}\psi^+_{\mathbf{y}t\omega_2}\psi^-_{\mathbf{y}t\omega_3} D^-_{\mathbf{x}t\mathbf{y}t\omega_4}:) \end{aligned} \tag{7.4}$$

To describe the action of the operator $\mathscr{L}_h$, we consider first the case when it operates on Wick monomials in the fields (7.1) of degree four.

In this case we define, if $\mathscr{R}_h \equiv 1 - \mathscr{L}_h$:

$$\begin{aligned} & \mathscr{L}_h:\psi^+_{x_1\omega_1}\psi^+_{x_2\omega_2}\psi^-_{x_3\omega_3}\psi^-_{x_4\omega_4}: \\ & \quad = 2^{-1}\sum_{j=1,2} :\psi^+_{x_j\omega_1}\psi^+_{x_j\omega_2}\psi^-_{x_j\omega_3}\psi^-_{x_j\omega_4}: \\ & \mathscr{R}_h:\psi^+_{x_1\omega_1}\psi^+_{x_2\omega_2}\psi^-_{x_3\omega_3}\psi^-_{x_4\omega_4}: \\ & \quad = \{:\psi^+_{x_1\omega_1} D^+_{x_2x_1\omega_2}\psi^-_{x_3\omega_3}\psi^-_{x_4\omega_4} \\ & \qquad + :\psi^+_{x_1\omega_1}\psi^+_{x_1\omega_2} D^-_{x_3x_1\omega_3}\psi^-_{x_4\omega_4}: + :\psi^+_{x_1\omega_1}\psi^+_{x_1\omega_2}\psi^-_{x_1\omega_3} D^-_{x_4x_1\omega_4}:\}_{\mathrm{AS}} \end{aligned} \tag{7.5}$$

where AS means antisymmetrization with respect to $(x_1, \boldsymbol{\omega}_1) \leftrightarrow (x_2, \boldsymbol{\omega}_2)$. The operator $\mathscr{L}$ is extended by linearity to the other monomials of fourth order. It turns out that $\mathscr{L}$ annihilates all the fourth-order monomials with more than one nonlocal field or with one nonlocal field. Otherwise

$$\begin{aligned} & \mathscr{R}:\psi^+_{x_1\omega_1} D^+_{x_2x_{2'}\omega_2}\psi^-_{x_3\omega_3}\psi^-_{x_4\omega_4}: \\ & \quad = :\psi^+_{x_1\omega_1} D^+_{x_2x_{2'}\omega_2}\psi^-_{x_3\omega_3}\psi^-_{x_4\omega_4}: \\ & \qquad - (:D^+_{x_2x_{2'}\omega_1}\psi^+_{x_2\omega_2}\psi^-_{x_2\omega_3}\psi^-_{x_2\omega_4}: + :\psi^+_{x_{2'}\omega_1} D^+_{x_2x_{2'}\omega_2}\psi^-_{x_2\omega_3}\psi^-_{x_2\omega_4}: \\ & \qquad + :\psi^+_{x_{2'}\omega_1}\psi^+_{x_{2'}\omega_2} D^-_{x_2x_{2'}\omega_3}\psi^-_{x_2\omega_4}: + :\psi^+_{x_{2'}\omega_1}\psi^+_{x_{2'}\omega_2}\psi^-_{x_{2'}\omega_3} D^-_{x_2x_{2'}\omega_4}:) \end{aligned} \tag{7.6}$$

$$\mathscr{L}{:}\psi^{+}_{x_1\omega_1} D^{+}_{x_2 x_{2'}\omega_2} \psi^{-}_{x_3\omega_3} \psi^{-}_{x_4\omega_4}{:}$$
$$= {:}\psi^{+}_{x_2\omega_1} \psi^{+}_{x_2\omega_2} \psi^{-}_{x_2\omega_3} \psi^{-}_{x_2\omega_4}{:} - {:}\psi^{+}_{x_{2'}\omega_1} \psi^{+}_{x_{2'}\omega_2} \psi^{-}_{x_{2'}\omega_3} \psi^{-}_{x_{2'}\omega_4}{:}$$

The action of $\mathscr{L}_h$ on the monomials of degree two is generated by

$$\mathscr{L}_h{:}\psi^{+}_{x_1\omega_1} \psi^{-}_{x_2\omega_2}{:} = {:}\psi^{+}_{x_1\omega_1}(\psi^{-}_{x_1\omega_2} + (x_2 - x_1)\, \mathscr{D}\psi^{-}_{x_1\omega_2}){:} \tag{7.7}$$

We now consider the second-degree monomials generated by multiplying $\psi^{+}_{x_1\omega_1}$ or $D^{+}_{x_1 x_2\omega_2}$ by a local field $\psi^{-}_{x_3\omega_2}$ or $\mathscr{D}\psi^{-}_{x_3\omega_2}$ or by any of the nonlocal fields of minus type in (7.1). The operator $\mathscr{L}_h$ transforms each of the above (12) monomials into a combination of the others. We write 1 for x_1, 2 for x_2, etc., and we shall not write explicitly the quasimomenta $\boldsymbol{\omega}_1$ of the plus-type fields and $\boldsymbol{\omega}_2$ of the minus-type fields. Then one easily finds that the action of $\mathscr{R}$ on the monomials on which it is not the identity and of $\mathscr{L}$ on the same monomials is

$$\mathscr{L}{:}\psi^{+}_1 \mathscr{D}\psi^{-}_2{:} = {:}\psi^{+}_1 \mathscr{D}\psi^{-}_1{:}$$
$$\mathscr{L}{:}\psi^{+}_1 D^{-}_{23}{:} = (x_2 - x_3){:}\psi^{+}_1 \mathscr{D}\psi^{-}_1{:}$$
$$\mathscr{L}{:}D^{+}_{12}\psi^{-}_3{:} = {:}\psi^{+}_1\psi^{-}_1{:} - {:}\psi^{+}_2\psi^{-}_2{:} + (x_3 - x_1){:}\psi^{+}_1 \mathscr{D}\psi^{-}_1{:} - (x_3 - x_2){:}\psi^{+}_2 \mathscr{D}\psi^{-}_2{:}$$
$$\mathscr{L}{:}D^{+}_{12}\mathscr{D}\psi^{-}_3{:} = {:}\psi^{+}_1 \mathscr{D}\psi^{-}_1{:} - {:}\psi^{+}_2 \mathscr{D}\psi^{-}_2{:}$$
$$\mathscr{L}{:}D^{+}_{12} D^{-}_{34}{:} = (x_3 - x_4)({:}\psi^{+}_1 \mathscr{D}\psi^{-}_1{:} - {:}\psi^{+}_2 \mathscr{D}\psi^{-}_2{:})$$
$$\mathscr{R}{:}\psi^{+}_1\psi^{-}_2{:} = {:}\psi^{+}_1 S^{1}_{21}{:} \tag{7.8}$$
$$\mathscr{R}{:}\psi^{+}_1 \mathscr{D}\psi^{-}_2{:} = {:}\psi^{+}_1 S^{2}_{21}{:}$$
$$\mathscr{R}{:}\psi^{+}_1 D^{-}_{23}{:} = {:}\psi^{+}_1 S^{1}_{23}{:} + {:}\psi^{+}_1 S^{3}_{3123}{:}$$
$$\mathscr{R}{:}D^{+}_{12}\psi^{-}_3{:} = {:}D^{+}_{12} S^{1}_{31}{:} + {:}\psi^{+}_2 S^{1}_{21}{:} + {:}\psi^{+}_2 S^{3}_{2132}{:}$$
$$\mathscr{R}{:}D^{+}_{12}\mathscr{D}\psi^{-}_3{:} = {:}D^{+}_{12} S^{2}_{32}{:} - {:}\psi^{+}_1 S^{2}_{12}{:}$$
$$\mathscr{R}{:}D^{+}_{12} D^{-}_{34}{:} = {:}D^{+}_{12} S^{1}_{34}{:} + {:}D^{+}_{12} S^{3}_{4134}{:} + {:}\psi^{-}_2 S^{3}_{2134}{:}$$

The above definition of the action of $\mathscr{L}_h$ on any V which is a sum of integrals of Wick monomials in the fields (7.1) produces term by term a result of the form

$$\int dx\, d\boldsymbol{\omega}_1\, d\boldsymbol{\omega}_2\, e^{ip_F(\omega_1 - \omega_2)x}[\nu(\boldsymbol{\omega}_1, \boldsymbol{\omega}_2){:}\psi^{+}_{x\omega_1}\psi^{-}_{x\omega_2}{:}$$
$$+ \alpha(\boldsymbol{\omega}_1, \boldsymbol{\omega}_2){:}\psi^{+}_{x\omega_1}\mathscr{D}\psi^{-}_{x\omega_2}{:} + \zeta(\boldsymbol{\omega}_1, \boldsymbol{\omega}_2){:}\psi^{+}_{x\omega_1}\mathscr{D}_t\psi^{-}_{x\omega_2}{:}]$$
$$+ \int e^{ip_F(\omega_1 + \omega_2 - \omega_3 - \omega_4)x}\, dx \prod_{i=1}^{4} d\boldsymbol{\omega}_i\, \lambda(\boldsymbol{\omega}_1, \boldsymbol{\omega}_2, \boldsymbol{\omega}_3, \boldsymbol{\omega}_4){:}\psi^{+}_{x\omega_1}\psi^{+}_{x\omega_2}\psi^{-}_{x\omega_3}\psi^{-}_{x\omega_4}{:} \tag{7.9}$$

where, of course, each term produces only one of the above addends.

The collection of all the contributions of the form (7.9) obtained by acting with $\mathscr{L}$ on $V^{(h)}$ will be written

$$V_L^{(h)} = \int e^{ip_F(\omega_1-\omega_2)x}\, dx\, d\omega_1\, d\omega_2\, (2^h \nu_h : \psi^{(\leqslant h)+}_{x\omega_1} \psi^{(\leqslant h)-}_{x\omega_2} : + \alpha_h : \psi^{(\leqslant h)+}_{x\omega_1} i\beta\omega_2 \mathscr{D}_{\omega_2} \psi^{(\leqslant h)-}_{x\omega_2} : + \zeta_h : \psi^{(\leqslant h)+}_{x\omega_1} \mathscr{D}_t \psi^{(\leqslant h)-}_{x\omega_2} :) - \int e^{ip_F(\omega_1+\omega_2-\omega_1'-\omega_2')x}\, dx \prod_{i=1}^{2} d\omega_i\, d\omega_i' \times \lambda_h(\omega_1, \omega_2, \omega_1', \omega_2') : \psi^{(\leqslant h)+}_{x\omega_1} \psi^{(\leqslant h)+}_{x\omega_2} \psi^{(\leqslant h)-}_{x\omega_1'} \psi^{(\leqslant h)-}_{x\omega_2'} : \tag{7.10}$$

It is important to notice that in (7.10) ν_h, α_h, and ζ_h are independent of ω_1, ω_2. This follows by observing that the effective potential could be calculated by doing the integration over the fields $\psi^{(n)}$, $n > h$, in a single step [see (4.12)], without introducing the fields $\psi^{\pm}_{x\omega}$. Of course also the dependence of λ_h on the ω's has to be somewhat special to reflect this *gauge invariance property*; see also Section 14.

Then the part of $V^{(h)}$ of degree two in the fields must have the form

$$\int dx\, dy\, w_h(x-y) : \psi^+_x \psi^-_y :$$

$$= \int dx\, dy\, d\omega_1\, d\omega_2\, e^{ip_F(\omega_1 x - \omega_2 y)} \times w_h(x-y) : \psi^{(\leqslant h)+}_{x\omega_1} [\psi^{(\leqslant h)-}_{x\omega_2} + (y-x)\, \mathscr{D}\psi^{(\leqslant h)-}_{x\omega_2} + S^1_{yx\omega_2}] : \tag{7.11}$$

with w_h being a suitable rotation-invariant distribution.

Acting with $\mathscr{L}_h$ on (7.11), one immediately finds the validity of the claim about α_h, ν_h, ζ_h.

The constants ν_h, α_h, ζ_h together with the function λ_h will be called *running form factors on scale h*. The ν_h will be said to have scaling dimension 1: and to remind us of this attribute, its definition contains the factor 2^h. The others will be said to have scaling dimension 0.

The result of the application of $\mathscr{R}_h$ on $V^{(h)}$ gives rise to an *irrelevant* part of $V^{(h)}$ defined by $V_R^{(h)} \equiv V^{(h)} - V_L^{(h)}$.

We shall denote for each $h \leqslant 0$ the running form factors as

$$\mathbf{v}_h = (\nu_h, \alpha_h, \zeta_h, \lambda_h) \tag{7.12}$$

and we do not call them running couplings, except if $d = 1$, because λ_h is a function; hence they correspond to infinitely many parameters per scale: the name *running coupling* usually denotes finitely many constants per scale.

To understand the meaning of the operations $\mathscr{L}$ and $\mathscr{R}$ and their relation with the heuristic considerations of Section 3, 5, we write the result of the action of $\mathscr{L}_h$ on the lhs of (7.11) as

$$\int [a_h \psi_x^{(\leqslant h)+} \psi_x^{(\leqslant h)-} + b_h \psi_x^{(\leqslant h)+} (-\Delta - p_F^2) \psi_x^{(\leqslant h)-}] \, dx \tag{7.13}$$

while the $1 - \mathscr{L}_h$ operation yields, for a suitable choice of the distribution $w'_h(x-y)$,

$$\int w'_h(x-y) \, \psi_x^{(\leqslant h)+} \psi_y^{(\leqslant h)-} \, dx \, dy \tag{7.14}$$

The distribution w'_h can be easily computed from (5.2), (7.5)–(7.8), and (7.11) and is trivially related to w_h; see (7.16) below.

Hence we see that the second-degree part of the effective potential on scale h can be written as

$$\int [a_h \psi_x^{(\leqslant h)+} \psi_x^{(\leqslant h)-} + b_h \psi_x^{(\leqslant h)+} (-\Delta + p_F^2) \psi_x^{(\leqslant h)-}] \, dx$$

$$+ \int w'_h(x-y) \, \psi_x^{(\leqslant h)+} \psi_y^{(\leqslant h)-} \, dx \, dy \tag{7.15}$$

The distribution $w'_h(x-y)$ has a Fourier transform at momentum $(k_0, \mathbf{k})$, which we write $\hat{w}'_h(k_0, p^2)$, with $p^2 = \mathbf{k}^2$, to take into account the rotational symmetry. It can be easily verified that

$$\hat{w}'_h(k_0, \mathbf{k}^2) = \hat{w}_h(k_0, \mathbf{k}^2) - \hat{w}_h(0, p_F^2)$$

$$- (\mathbf{k}^2 - p_F^2) \frac{\partial \hat{w}_h}{\partial p^2}(0, p_F^2) - k_0 \frac{\partial \hat{w}_h}{\partial k_0}(0, p_F^2) \tag{7.16}$$

We see, recalling (3.11) and (5.27), that, in order to check the existence of the Fermi surface, we have to verify the existence of $w_h(x-y)$ and a fast enough decay at ∞: what is needed is that $\hat{w}_h(k_0, \mathbf{k}^2)$ is so smooth that, near the Fermi surface $k_0 = 0$, $|\mathbf{k}| = p_F$, the rhs of (7.16) goes to zero faster than $|k_0| + |\mathbf{k}^2 - p_F^2|$. See Section 11 for a further discussion of this point.

It will be clear that (7.5)–(7.8) are *overdoing* some subtractions. In fact, the basic bounds of Section 10 would work if instead of insisting that $\mathscr{L}$ and $\mathscr{R}$ be linear operators, we just defined them as operations whose action on a linear combination of Wick monomials is defined by acting in a prescribed way on each of the monomials and taking then the same linear

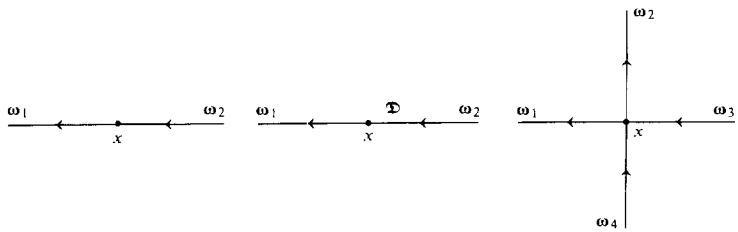
combination. The latter procedure would be possible provided the effective potential expression in terms of Wick monomials is produced according to well-defined rules (as will be our case). The results would, of course, depend on the particular path followed in the construction in case the latter contains arbitrary choices. The price we would pay would be that we could no longer be *a priori* sure that the gauge invariance property introduced after (7.10) holds.

The advantage would be a simplification in the structure of $\mathscr{L}$ and $\mathscr{R}$: for instance, we could simply define $\mathscr{L}$ be (7.5) on a product of four fields $\psi^{\pm}$ and set the $\mathscr{L}$ in (7.6) simply equal to 0, i.e., set $\mathscr{L}$ to zero when acting on fourth-degree monomials with dimension $\delta_P \geqslant 1$; similarly, we could set $\mathscr{L}$ equal to zero on the terms of (7.8) with dimension $\delta_P \geqslant 1$ and we could set $\mathscr{L}(:D_{12}^{+}\psi_3^{-}:) = (x_1 - x_2):\mathscr{D}\psi_3^{+}\psi_3^{-}:$ rather than using the more complicated expression in (7.8). The bounds in Section 10 would work without change.

In Section 11 we adopt partially this viewpoint, i.e., we modify the definition of $\mathscr{L}$, giving up the linearity by setting it equal to zero in the cases (7.6); but we shall stick to the basic definition (7.7) because giving up the gauge invariance in the terms of second degree would, in the long run, produce disadvantages which would become overwhelming. It might be interesting to know whether the ideas of ref. 34 could be useful in this context.

8. GRAPHICAL INTERPRETATION

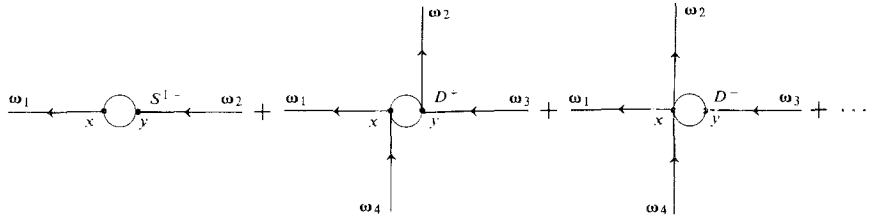
We start by representing graphically the various terms of the relevant interaction V_L^0 in (6.6):



(8.1)

representing, respectively, also the three integrands in the rhs of (7.10) (we say that they are three and not four, because $\zeta_0 = 0$). The superscripts remind us of the meaning of the lines as fields. We call the three graph elements in (8.1) the *local* graph elements.

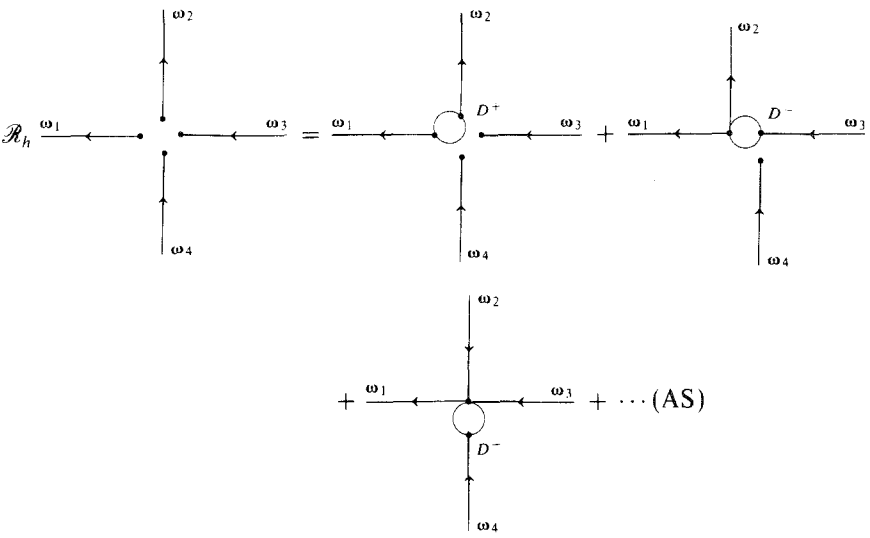
The irrelevant part of the interaction V_R^0 [see (7.4)] will be represented similarly:



(8.2)

with a self-explanatory notation, allowing us to identify unambiguously each term in (7.4); the dots refer to the graphs representing the last two terms in (7.4). The circle denotes the fact that the nonlocal field D or S^1 has as indices the two points which it joins in the graph. The nonlocal fields D and S^1 are regarded as emerging from the point corresponding to their first label, i.e., the point in the picture from which the line labeled D or S^1 emerges.

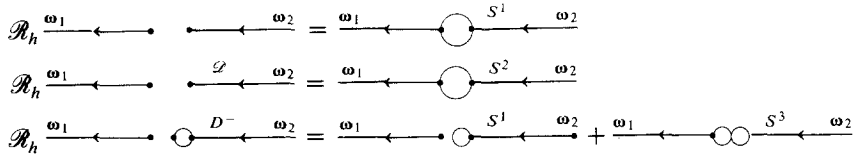
We can find also a natural representation for the action of the $(1 - \mathscr{L}_h)$ operators; in fact (7.5) yields the following picture:



(8.3)

where the dots refer to the three similar graphs representing the other

terms in (7.5), due to the AS operation. Moreover, the relations (7.8) can be represented as



(8.4)

In similar way we can represent the action of $\mathscr{R}_h$ on the more complicate second-degree monomials involving D^+ in (7.8) or the action of $\mathscr{R}_h$ on the fourth-degree monomials (7.6). We do not report the corresponding diagrams, which the reader can easily imagine.

Finally, we can represent the relevant part of $V^{(h)}$ given in (7.10) by (8.1) if $\mathscr{D}$ over a line means either $\mathscr{D}_t$ or $i\boldsymbol{\omega}\cdot\mathscr{D}_{\boldsymbol{\omega}}$. This unification in the notation is very useful, as it simplifies considerably the graphical representations.

9. THE BETA FUNCTIONAL

The beta functional is defined to be a function B_h such that

$$\mathbf{v}_{h-1} = A\mathbf{v}_h + B_h(\mathbf{v}_h, \mathbf{v}_{h+1}, \ldots, \mathbf{v}_0) \tag{9.1}$$

where A is a suitable linear transformation trivially operating as a multiplication by a suitable constant (1 or 2) on the four *running form factors* of the relevant part of $V^{(h)}$ [see (7.9)]: $\mathbf{v}_h = (\nu_h, \alpha_h, \zeta_h, \lambda_h)$.

We apply the methods developed in refs. 16–18 to study the functional B. We assume that the reader is familiar with the structure of the tree expansion.[17] We refer to the literature for the motivation behind the definitions below.[18,16]

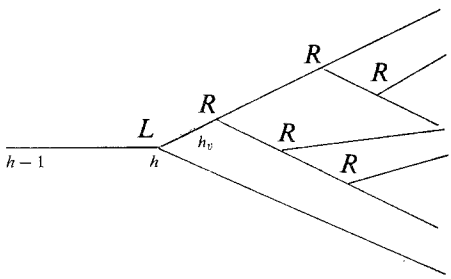
The tree expansion provides the recursive expression (9.1), i.e., essentially an expansion for the relevant part $\mathscr{L}_h V^{(h)}$ of the effective potential on scale h together with an expansion of the remaining irrelevant part $\mathscr{R}_h V^{(h)}$ in terms of the form factors $\mathbf{v}_{h+1}, \ldots, \mathbf{v}_0$.

It is obvious from the definitions of $\mathscr{L}$, $\mathscr{R}$ of Section 7 that this is possible: in fact, this is a purely formal statement and it holds whether we consider the definition (7.6)–(7.8) of $\mathscr{L}$ or the modification suggested at the end of Section 7 or the other modification introduced later in Section 11; see (11.7). Below we only describe the result for the first definition of $\mathscr{L}$ in Section 7: its proof is inductive. We skip all motivation steps, as the formalism is identical to the one introduced in refs. 16 and 17 and in ref. 18

in the case of the bosons, and we wish to avoid a very long repetition. Nevertheless, we stress that in principle what is described here is self-contained. If one has enough faith to read the definition of the rhs of (9.1) (to which this section is completely devoted), then it is very easy to check by induction that (9.1) is in fact correct with our expression for the rhs. The adaptation of what follows to other definitions of $\mathscr{L}$ is immediate.

Let ϑ be a tree with m endlines, and let v be a vertex of ϑ bearing a *frequency* or *scale* label h_v.

The beta function is associated with trees like



(9.2)

in which the first nontrivial vertex has frequency index h and bears an L-label (meaning that the operation $\mathscr{L}_{h-1}$ has been applied to the function of $\psi^{(h-1)}$ symbolized by the tree without the label L). All the other tree vertices v carry a label R, meaning that the operations $\mathscr{R}_{h_v-1} = 1 - \mathscr{L}_{h_v-1}$ have been applied (see below).

The trees form a partially ordered set of vertices where the root is the vertex to which the index $h-1$ is attached: the frequency indices strictly increase as one moves monotonically from the root toward the endlines. In practice we write trees as in (9.2) and orient them from left to right.

If ϑ has m endlines, we label then $1, 2, \ldots, m$ from top to bottom: we identify trees which, before the labeling, are topologically identical and we imagine selecting with some rule one representative tree per equivalence class, never drawing the others. The labeling is also not arbitrarily set down, but we label the points from top to bottom (say): hence one can estimate that there are only $< 2^{4m}$ trees, in the above sense, with m endlines.

Each vertex $v \in \vartheta$ can be thought of as the first nontrivial vertex of the subtree $\vartheta_v \subset \vartheta$ with root at the vertex v' preceding v in the ordering of ϑ. The m_v endlines of the tree ϑ define a subset of the set of endline indices which we call the *cluster associated with* v, or the *cluster* v. So the clusters with higher frequency have smaller size, and if v' precedes v in the tree partial ordering, then v' is actually larger in size; i.e., the tree ordering is, in

terms of clusters, according to decreasing size or scale and to increasing frequency.

Our purpose is to introduce the notion of scaling decomposition of a Feynman graph: it will be described in terms of *ordinary* Feynman graphs carrying a rather large number of extra labels which are meant as reminders that the calculations of their values are suitably modified with respect to the usual way of computing undecomposed graphs. We shall call the new objects simply *Feynman graphs* to avoid a proliferation of names.

To each tree ϑ one can associate a set of ϑ-compatible graphs. This is done, as already suggested by the graphical representations of Section 8, as follows.

1. Associate with each endline $1, 2, \ldots, m$ of ϑ one of the graph elements introduced in (8.1) and (8.2), appending an extra label $1, 2, \ldots, m$ to all the space-time labels already appearing in them: this is done in order to identify to which of the m endlines they have actually been associated. We call the vertices in (8.1) *relevant vertices*, because they correspond to relevant operators in the sense of Section 7. A cluster of endlines is thus also a cluster of vertices and hence of space-time points. In this way we can think of the clusters associated with the tree vertices also as clusters of space-time points ordered hierarchically by inclusion.

2. Connect some (up to all but two) of the lines in pairs, allowing a pair of lines to form a single line only if the directions of their arrows do not come into conflict. We thus form a *graph G*.

3. For each vertex $v \in \vartheta$ we can consider, as mentioned in step 1, the set of space-time indices of the points in the graph elements associated with the m_v endlines of the *cluster v*: thus we can think of the cluster of points in v as a cluster of space-time points which we call the *v-cluster*.

We enclose into an ideal box the v-cluster together with all the lines of the graph G which join pairs of points in v: in this way we isolate naturally a subgraph G_v of G whose internal lines are all inside the box of the v cluster. It is the graph obtained from G by cutting in half every line in G which joins a point of the v-cluster to points outside the v-cluster and then deleting the part of G disconnected from the points lying inside the cluster v; i.e., we delete all the graph elements not *belonging* to the cluster v.

4. We append to each inner line an index s or h classing it as a *soft* or a *hard* line. If a line λ is inner to a cluster v but not to any smaller one, then we say that h_v is the *scale* or *frequency index* of λ. If the line λ has one endpoint inside a cluster v_1 and the other in v_2 but not in v_1, with v_2 larger than v_1, then we say that λ *crosses* v_1: hence a line crosses all clusters which contain one endpoint and not the other.

5. We discard G unless for all vertices $v \in \vartheta$ the subgraph G_v is connected and the connection can be realized by considering only the hard lines inner to G_v.

We say that G is compatible with ϑ if it survives step 5: note that the entire graphical discussion just presented translates some rather simple and natural geometrical observations and conditions.[16-18]

In words, a graph G is compatible with ϑ if it is the union of a collection of connected subgraphs, hierarchically ordered by inclusion, so that the inclusion relation provides a realization of the partial order structure of the tree ϑ, and furthermore the connection properties of G do not change if only the hard lines are taken into account. The reader should try to make a few drawings and examples for a better understanding of the above notions.

The above definitions introduce graphs G with internal and external lines. Each internal line arises by joining together two lines of the basic graph elements.

Hence we shall think of the external lines as *half lines* and of the internal lines as *full lines* composed of two half lines uniquely identified by the graph elements from which they come.

Then we look at all subgraphs G_v of G, supposed ϑ-compatible, which have two or four external lines. If v is a tree vertex carrying an R superscript, we shall add, according to a rule that we are about to describe, superscripts D or S to some of the half lines emerging from G_v.

This is supposed to reflect the application of the operator $\mathscr{R}_{h_v-1}$ to the function of $\psi^{(\leq h_v-1)}$ arising from computation of the truncated expectation symbolized by the tree vertex v.

This is done as follows, and is based on the graphical representation of the operation $\mathscr{R}_{h_v}$ described in Section 8.

(a) When G_v has four external lines of $\psi^{\pm}$ type, the $\mathscr{R}_{h_v}$ operation can be simply thought of as replacing G_v with a new graph [chosen out of up to six as shown in (8.3)] identical to G_v except for one of the external half lines, which is replaced by a D line, and some other external half lines are thought of as emerging from different space-time points of the same cluster v. The actual number of choices of modified graphs could be smaller than 6 when some of the space-time points coincide (in which case some D field may vanish). We distinguish the choices of the various graphs by adding a label to the cluster v (say, a number from 1 to 6).

(b) When G_v has two external lines, the action of $\mathscr{R}_h$ also simply amounts to replacing G_v by a new graph (chosen out of up to three possibilities), as (8.4) shows. We distinguish the various graphs by adding a label to the cluster v (say, a number from 1 to 3).

This completes the description of the meaning of the R labels on the tree vertices. It remains to explain the L label on the first nontrivial vertex.

The presence of such a label simply means that we discard all graphs G with more than four external lines. For the others we replace G_{v_0} by a few new graphs obtained from G_{v_0} by changing the meaning of the external lines according to a prescription similar to the one used for $\mathscr{R}$, but using, if one wants a graphical representation, an appropriate graphical representation of the $\mathscr{L}$ operator similar to the ones of Section 8. Each G_{v_0} is thus replaced by up to four terms; all of them are *local*, i.e., they represent monomials P of the form $:\psi^+_{x\omega_1}\psi^+_{x\omega_2}\psi^-_{x\omega_3}\psi^-_{x\omega_4}:$, $:\psi^+_{x\omega_1}\psi^-_{x\omega_2}:$, or $:\psi^+_{x\omega_1}\partial\psi^-_{x\omega_2}:$.

The collection of the tree ϑ, the graph G, and of all the labels added to it in the above construction will be called a *relevant Feynman graph.*

A similar construction can be performed also for trees like the one in (9.2), which however, carry an R label on the first nontrivial vertex and a scale index h_{v_0} not necessarily equal to h (but $\geqslant h$). One just treats the cluster v_0 as already done for the inner ones. The collection of the tree ϑ, the graph G, and of all the labels added to it in the above construction will be called in this case an *irrelevant Feynman graph.*

Each of the above graphs [there are $O((2m)!)$ of them associated with each of the trees with m endlines] will be given a *value* which is a number depending on all the indices appended to the graph.

Postponing the description of the rules to construct the value $w_{h,G}$, we associate with each graph:

I. An *operator* O_P, which is a contribution to the effective potential on scale $h-1$, and

II. A *size*, which is a positive number.

The operator O_P is simply obtained by multiplying the graph value with the Wick monomial P formed by the product (in a suitable order) of the fields symbolized by the external lines of the graph and subsequently integrating the resulting expression over the space-time points and over the quasiparticle momenta associated with the fields forming P, and summing over the frequency indices. In other words, O_G is the operator with form factor equal to the graph value, and it is an element of the Grassmanian algebra generated by the quasiparticle fields (hence it is somewhat improper to call it an operator: we follow here the traditional terminology).

The *size* is obtained by considering the graph value $w_{h,G}$ and the integrals

$$C_N = \sup_{\substack{k_2,\dots,k_{2n} \\ \omega_1,\dots,\omega_{2n} \\ i_1,\dots,i_{2P}}} \left| \int \left[\sum_{i,j} 2^{2h}(x_i - x_j)^2 \right]^N w_{h,G}(x_1,\dots,x_{2n},\omega_1,\dots,\omega_{2n}) \right.$$

$$\times \left(\exp \left\{ i \sum_{j=1}^{n} [k_j(x_j - x_1) - k_{j+n}(x_{j+n} - x_1)] \right\} \right)$$

$$\times 2^{\delta_P h} (d_P)_{i_1, \ldots, i_{z_P}} \prod_{j=2}^{2n} dx_j \Bigg| \tag{9.3}$$

where x_1 is the localization point, $x_1, \ldots, x_n$ are the points out of which emerge the external lines of plus type, and $x_{n+1}, \ldots, x_{2n}$ are the points from which emerge the minus-type lines; if some of the external lines are nonlocal fields, one has to integrate also over their space-time labels, which, however, have not been explicitly introduced in (9.3), for simplicity. The quantity δ_P is the dimension of the operator associated with the graph, and d_P is the tensor describing the order of zero of the nonlocal fields; see (7.3). Here N is a pre-fixed parameter: when we discuss the N dependence we shall refer to (9.3) as defining the *C_N-size* of the graph. In most cases we need only $N = 0$.

If the graph represents a local operator (i.e., an operator with only one vertex attached to external lines), the above size involves no integral: this is the case for the graphs contributing to the beta functional (see below) and motivates the introduction of the concept.

The *value* of each of the above graphs is defined as the integral of a product of various *factors* over the internal line quasimomenta and the inner space-time points (i.e., the points not appearing in the external fields): each pair of half lines forming a full line will contribute to the result a factor equal to the covariance of the two fields symbolized by the two lines. The covariance will be *soft* or *hard*, depending on the label on the line. If h is the frequency index of the line, we use $g^{(<h)}$ for soft lines and $g^{(h)}$ for hard lines.

One attributes a phase factor to the above expression: one counts the parity of a permutation that it is necessary to perform on a set of anticommuting Grassmanian fields each of which symbolizes a half line of the graph elements of Section 8 corresponding to the endpoints of the tree. We imagine them written next to each other on a row according to the ordering inherited from the labels 1, 2,..., m. The Grassmanian fields representing lines corresponding to the same graph element will be written down in the same order in which they appear in (6.6) and (7.4), i.e., looking at (8.1) and (8.2), counterclockwise (say) starting from the left. Then we permute the Grassmanian fields by carrying next to each other every pair of Grassmanian fields symbolizing two half lines forming a full line in the graph, with the object corresponding to the field with index minus to the left. The noncontracted or *external* lines correspond to Grassmanian fields which are not paired in the given graph: we nevertheless bring them, too,

next to each other with the plus fields to the left of the minus fields and the sign in question will be ± 1, depending on the evenness or oddness of the total permutation just described. The Wick product of the fields associated with the external lines, taken in the order reached after the above permutations, defines the Wick monomial P of the graph. It is the Grassmanian variable to be used to construct the operator associated with the graph.

A combinatorial factor $1/s_v!$ is associated with every vertex v if s_v is the number of tree branches following v.

The construction of the value continues by looking at the ω indices of the graphs lines and adding to the list of the factors so far associated with the graph a factor $\exp i(\sum \pm p_F \omega_j \mathbf{x}_j)$ if x_j is the space-time point from which field lines (up to four) with quasimomenta ω_j emerge. The sign choice is plus for the lines oriented out of x_j and minus for the ones oriented toward x_j.

We also multiply by a factor equal to the product of the form factors $\mathbf{v}_{h_j}$ corresponding to the graph elements described by the tree endpoints: each form factor is computed at the frequency h_j to which the endpoint is linked to the tree ϑ by the respective endlines of ϑ. A factor 2^{h_j} is put in front of every v_{h_j}.

Once all the above operations have been performed, we have to distinguish if we are computing the value of a relevant graph or that of an irrelevant graph. In the latter case we just multiply all the factors and we obtain in this way a function of the graph labels, which we call the value of the graph, after integration over the inner points and the inner quasiparticle momenta.

If the first vertex bears an L label as in (9.1), which is the case if one wants to study the beta function, then we integrate over all the space-time coordinates of the graph vertices other than the localization point of the graph and over the quasimomenta of the inner lines. We are left, in this case, with an expression which, once multiplied by the fields representing the external lines (in the order found when computing the overall sign) and integrated over the localization point and over the quasimomenta of the external lines, looks like one of the terms in (7.9) which is the contribution to the $V_L^{(h)}$ of the given tree and graph.

Therefore the tree in (9.1) defines a set of contributions to $V^{(h-1)}$ which add up to an expression like (7.10) with $h-1$ replacing h and defining what will be called $V_L^{(h-1)}$ and the corresponding form factors $\mathbf{v}_{h-1}$ expressed in this way (by construction) as functions of $\mathbf{v}_h, \mathbf{v}_{h-1}, \ldots, \mathbf{v}_0$. This also completes the description of the *beta functional* $A + B_h$ in (9.1); and in fact the linear part A is simply

$$A\mathbf{v}_h = A(\nu_h, \alpha_h, \zeta_h, \lambda_h) = (2\nu_h, \alpha_h, \zeta_h, \lambda_h) \tag{9.4}$$

Following the standard terminology, we say that the coupling v_h is *relevant*, while the others are *marginal*; also we say that v_h has *dimension* 1 and α_h, ζ_h, λ_h have dimension 0.

It is the main result of the formalism introduced in refs. 16–18 that the effective potential on scale $h-1$ is simply

$$V^{(h-1)} = V_L^{(h-1)} + V_R^{(h-1)} \tag{9.5}$$

where $V_L^{(h-1)}$ is the sum of the operators associated with the relevant graphs and $V_R^{(h-1)}$ is the sum of the operators associated with the irrelevant ones. The identity (9.5) is not deep, and it is easily proved by induction: of course, it is hard to work out the description of the result given above if one has no familiarity with the tree expansion: however, to present here the above material in a deductive style would be a word by word repetition of the work in refs. 16–18 and the reader is referred there (e.g., ref. 17, Sections 16–20).

To shorten the notation, we introduce a special symbol E_{ϑ} to denote the combinatorial factor always present in the graph value:

$$E_{\vartheta} = \prod_v \frac{1}{s_v!} \tag{9.6}$$

The reader may think that the above decomposition of $V^{(h-1)}$ into relevant and irrelevant parts is *arbitrary*. The nontriviality of the above formulation is that if one supposes $|\mathbf{v}_h| \leqslant \varepsilon$ for all h [see (7.12)] and looks at the mth-order terms in the variables $\mathbf{v}_{h+1}, \ldots, \mathbf{v}_0$, then one finds, for suitably chosen D, C, that their sum can be bounded by the h-independent quantity $m!\, C^{m-1} D \varepsilon^m$. And if the *size* of the operators contributing to the effective potentials is defined as above, also the sum of the sizes of the mth-order contributions from the various diagrams is bounded by the same quantity (independent of h).

By the definition (9.3), the size C_0 of the relevant contributions to the effective potential, i.e., the contributions to the beta functional, from graphs of given order m evaluated on a given tree ϑ is a bound on the absolute value of the corresponding contribution to the mth-order coefficients of the formal power series defining the beta functional (9.1): hence a perturbation theory of the beta functional will be a statement about a bound like DC^{m-1} from such coefficients.

The latter statement is interesting because it shows that we have not really overdecomposed a big number as sum of many small ones: essentially it shows that for the purpose of the bounds on the beta function and the form factors we have bounded each graph of order m by C^m with C graph independent: this is the best one can hope. *It is also the main result which is proved in Section 10.*

If one modifies the $\mathscr{L}$, $\mathscr{R}$ operations in a way which still can be algebraically represented by formulas like (7.5)–(7.8) (e.g., in one of the ways discussed in the last remark of Section 7), it is clear that the above description would be modified only in the labeling procedure. We shall make use of this remark in Section 11 and again in Section 14.

10. ESTIMATES FOR THE BETA FUNCTIONAL

We consider a general graph G compatible with a given tree ϑ, with root frequency $h-1$.

Let m_1, m_2, m_3 be the numbers of graph elements of the three types in (8.1). We consider first the case in which there are, in G, only relevant vertices, i.e., none of the types in (8.2): the latter will not be considered explicitly and will never pose extra problems. In fact, the renormalization transformation produces them anyway already in $V^{(-1)}$, together with many other irrelevant terms. Hence it is not really restrictive to assume that they are initially absent, but it simplifies a little the formal aspects of the analysis. Thus the graph G will contain $m = m_1 + m_2 + m_3$ vertices and

$$\begin{aligned} 2m_1 + m_2 + 4m_3 &\quad \text{half lines of type } \psi^{\pm} \\ m_2 &\quad \text{half lines of type } \partial\psi^{\pm} \end{aligned} \tag{10.1}$$

where the type of a half line considered here is its type in the original elements of the graph; recall that after all the labels have been appended to the graph some of the lines may change type [see Section 9, comments (a), (b)].

If G has $2n$ external lines of frequency h and of type ψ, its contribution on the tree ϑ to the effective potential $V^{(h)}$ has, before the action of the localization operators $\mathscr{L}$ and $\mathscr{R}$, the form

$$\begin{aligned} \int \left\{ \exp\left[i p_F \left(\sum_{j=1}^{n} (\boldsymbol{\omega}_j \mathbf{x}_j - \boldsymbol{\omega}_{n+j} \mathbf{x}_{n+j}) \right) \right] \right\} \\ \times W_h(x_1, \ldots, x_{n+1}, \ldots, x_{2n}, \boldsymbol{\omega}_1, \ldots, \boldsymbol{\omega}_{2n}) \\ \times :\psi^{+}_{x_1 \boldsymbol{\omega}_1} \cdots \psi^{-}_{x_{2n} \boldsymbol{\omega}_{2n}}: \prod_{j=1}^{2n} dx_j \, d\boldsymbol{\omega}_j \end{aligned} \tag{10.2}$$

and W_h may contain some δ-functions: we allow this singularity to compactify the notation.

Hence, for $2n=4$, for instance, the contribution of this graph to the beta functional will be [see the first of (7.5) and (7.9)]

$$\int \exp\{ip_F[\omega_2(\mathbf{x}_2-\mathbf{x}_1)-\omega_3(\mathbf{x}_3-\mathbf{x}_1)-\omega_4(\mathbf{x}_4-\mathbf{x}_1)]\}$$

$$\times W_h(x_1, x_2, x_3, x_4, \omega_1, \omega_2, \omega_3, \omega_4) \prod_{j=2}^{4} dx_j \tag{10.3}$$

Similar expressions with fewer variables and integrals arise when considering the other graphs, relevant for the beta function, with two external lines like (10.2) of the types $\psi^+\psi^-$ or $\psi^+\mathscr{D}\psi^-$.

It is useful to remark here, to avoid repetitions, that even the other (irrelevant) graphs lead to contributions like (10.2) to the effective potential: one just has to consider that some of the fields ψ are replaced by the nonlocal fields S^1, S^2, S^3, $D^{\pm}$; in the latter cases the corresponding space-time indices have to be thought of as pairs or triples of space-time indices.

In this section we shall often use the locution *integrating over some of the graph labels*: by this we mean integrating with respect to the space-time labels $x=(\mathbf{x}, t)$ and to the quasimomentum labels ω and summing over the discrete indices such as the frequency indices or the other indices appended to the graphs in the construction of Section 9.

We want to bound the *size* of a graph G, (see Section 9, item II); for simplicity, we shall consider first the case of a graph relevant for the beta function for the running form factor λ_h. We have to estimate an expression like (10.3), which we write more explicitly as

$$E_{\mathcal{G}} \int \exp\{ip_F[\omega_2(\mathbf{x}_2-\mathbf{x}_1)-\omega_3(\mathbf{x}_3-\mathbf{x}_1)-\omega_4(\mathbf{x}_4-\mathbf{x}_1)]\}$$

$$\times \prod_i [\lambda_{h_i}(\mathbf{\Omega}_i)\, 2^{\varepsilon_i h_i}]\, dx_2 \cdots dx_m$$

$$\times \left\{ \prod_{\substack{l \text{ inner} \\ \text{line}}} d\omega_l\, [\exp(ip_F \omega_l \cdot \xi_l)]\, 2^{\eta_l h_l} \right.$$

$$\left. \times g_l(2^{h_l}\xi_l^{(1)}, \omega_{1,l}; \ldots; 2^{h_l}\xi_l^{(j_l)}, \omega_{j_l,l}) \right\} \tag{10.4}$$

where $E_{\mathcal{G}}$ is the combinatorial factor (9.6), the $\mathbf{\Omega}_i$ are the ω's involved in the ith endline graph element, and $\varepsilon_i=0, 1$, depending on the *dimension* (see Section 9) of the coupling constant of the ith graph element [$\varepsilon_i=1$ only for the m_1 graph elements of the first two of the types in (8.1)]; the

h_i is the frequency index of the vertex to which the ith endline is attached. Moreover, $2^{\eta_l h_l} g_l$ is the propagator of the two fields represented by the two half lines forming the line l and $\xi_l = x_j - x_i$ if l is an oriented line joining x_i to x_j; g_l depends on $j_l \leqslant 4$ differences of vertex coordinates $\xi_l^{(1)}, \ldots, \xi_l^{(j_l)}$, which are determined from the types of the two fields and from the choice of the localization monomials in the rhs of the appropriate relation in (7.5)–(7.8): note, however, that the $\xi_l^{(i)}$ are always differences between space-time coordinates of points in the same cluster associated with the $\mathscr{R}$ operation that last changed the meaning of l before its contraction. Finally, η_l is the sum of two contributions each coming from one of the two half lines forming l: a half line of *type* 0 (i.e., line ψ, D, S^1) contributes 1/2, while a half line of *type* 1 (i.e., a line $\mathscr{D}\psi$, S^2, S^3) contributes 3/2. In this way the g_l functions are $O(1)$ as $h \to -\infty$ [see (4.3), (4.5)].

Let, in general,

$$
\begin{aligned}
n^i_{0v} &= (\text{number of half lines of type } \psi^{\pm}, D, S^1 \text{ inner to the cluster } v, \\
&\qquad \text{but not to smaller clusters}) \\
n^i_{1v} &= (\text{number of half lines of type } \mathscr{D}, S^2, S^3 \text{ inner to the cluster } v, \\
&\qquad \text{but not to smaller clusters}) \\
n^e_{0v} &= (\text{number of half lines of type } \psi^{\pm}, D, S^1 \text{ going out of the cluster } v) \\
n^e_{1v} &= (\text{number of half lines of type } \mathscr{D}\psi^{\pm}, S^2, S^3 \text{ going out of the} \\
&\qquad \text{cluster } v) \\
s_v &= (\text{number of tree branches emerging from the vertex } v) \qquad (10.5) \\
v_1, \ldots, v_{s_v} &= (\text{tree vertices immediately following } v) \\
h_i &= (\text{frequency index of the vertex to which the } i\text{th endline is attached}) \\
\mathbf{\Omega}_v &= (\text{set of the quasiparticle momenta corresponding to the lines} \\
&\qquad \text{emerging from the cluster } v)
\end{aligned}
$$

where the label v sometimes will be omitted when $v = v_0$.

We remark that the integration over the ω's corresponding to the internal lines could be explicitly performed [using (4.8)]. However, it is convenient to perform first the integration over the x variables and estimate the result before performing the ω integrations.

We split the following discussion into two parts. We first suppose that the $\mathscr{R}$ operation has no effect on the lines of G: in this case only half lines of type ψ, $\mathscr{D}\psi$ are present [recall that we are assuming that the graph G is built only of the relevant graph elements (8.1)].

The second case will be when we consider graphs in which the $\mathscr{R}$ operation has produced nontrivial effects, so that the graph may also con-

tain some lines of the types D, S^i, i.e., lines reflecting a space-time localization.

In the first of the above cases, given ϑ, G, let us draw a spanning tree μ inside G consisting of internal hard lines and such that its part inside the clusters G_v of G, associated with the vertices of ϑ, is also a tree μ_v of hard lines; we take μ to be *spanning*, i.e., we suppose that μ connects all vertices. We define the root of μ to be the localization point of the graph G. Then, for every v, the subtree μ_v will also be a subgraph of G_v.

We compute the integrals of expressions like the one in (10.4) via a Fourier transform in which we express every propagator by means of its Fourier integral. If we consider, for the purpose of unifying the notations, that ω_l defines a $(d+1)$-vector $\omega_l = (0, \boldsymbol{\omega}_l)$ with a vanishing time component, then we see that every inner line λ will carry a momentum $p_F \omega_l + 2^{h_l} k_l$ and a propagator $2^{\eta_l h_l} \hat{g}_l(k_l, \omega_l)$, and the x integrations are replaced by k_l integrations. To shorten the notations, we write $\hat{g}_l(k)$, $g_l(x)$ instead of $\hat{g}_l(k, \omega)$, $g_l(x, \omega)$, unless the more precise notation is required.

Then the x integrations are replaced by k_l integrations and each vertex of the graph will provide a $(d+1)$-dimensional momentum conservation δ function [which are ordinary $(d+1)$-dimensional Dirac distributions] with the exception of the localization vertex with label x_1:

$$\prod_{\substack{\text{all vertices } x \\ \text{but } x_1}} \delta\left(\sum_{\substack{\text{lines } l \\ \text{converging in } x}} \pm (2^{h_l} k_l + \omega_l) \right) \prod_{l \in \text{hard}} 2^{\eta_l h_l} \hat{g}_h(k_l)$$

$$\times \prod_{l \in \text{soft}} 2^{\eta_l h_l} \hat{g}_s(k_l) \prod_{\text{endlines}} \lambda_{h_i}(\boldsymbol{\Omega}_i)\, 2^{\varepsilon_i h_i} \tag{10.6}$$

where $k_l = 0$ if l is an external line and the $\pm$ sign depends on whether the line l is incoming or outcoming and $\hat{g}_h(k)$ essentially decays at ∞ as $e^{-\kappa |k|}$, and is bounded everywhere, while $\hat{g}_s(k)$ is bounded by

$$|\hat{g}_s(k, \omega)| \leqslant \hat{C} e^{-\kappa |k|} / k^d \tag{10.7}$$

where the singularity at the origin comes from the fact that [see (4.5), (4.11), (A.6)]

$$g_s(x, \boldsymbol{\omega}) = \sum_{p=-\infty}^{-1} 2^p g_{p+h_\lambda}(2^p x, \boldsymbol{\omega}) \tag{10.8}$$

and the decay rate κ is h_λ independent (see Section 4 and Appendix A).

We proceed to evaluate the δ-function integrals by using them to compute the momenta of the hard lines: we need one δ function per hard line and we get a volume factor $2^{-(d+1)h_l}$ per vertex if h_l is the scale of the line

l of lowest frequency among those connecting the vertex to the rest of the spanning tree μ; the number of vertices contributing the factor $2^{-(d+1)h_l}$ equals the number $n-1$ of lines of the spanning tree. After this has been done the argument of the propagator corresponding to a hard line in the tree μ will have the form $2^{-h_l}p_F\omega_l + p$, where p is a suitable linear combination of the other graph momenta.

Supposing that the λ_h functions together with ν_h, α_h, ζ_h are bounded uniformly in h, we shall bound them, when convenient, by a constant M. We can then estimate the integrals over the ω_λ starting from those corresponding to the most external lines of the spanning tree μ and proceeding toward those corresponding to the other lines. Each integration can be bounded by $C\cdot 2^{(d-1)h_\lambda}$ because the $g_h(2^{-h_\lambda}p_F\omega_\lambda + p)$ multiplied by $2^{-(d-1)h_\lambda}$ produces an approximate δ function on the sphere if C is a suitably chosen constant (and in fact if $|p| \neq 2^{-h_l}p_F$, the integrals are much smaller: a basic fact that will have to be used later). This is correct even in the limiting case $d=1$, when the ω integration is just a sum over $\omega = \pm 1$.

It remains to perform the $(d+1)$-dimensional integrals over the soft line momenta or over the hard line momenta corresponding to lines which are not in the selected spanning tree μ. To estimate the integral, one has to remark that the momenta which are left after the above integrations are free integration variables (they are basically the loop momenta) because the tree μ was a spanning tree, and the propagators are bounded by a constant times either $e^{-\kappa|k|}$ or $e^{-\kappa|k|}/k^d$, which are summable functions [since the integrals over k are $(d+1)$-dimensional integrals].

Hence the whole integration process gives a result which can be bounded by a constant per line times a factor $2^{[-(d+1)+(d-1)]h_\lambda}$ per line of the spanning tree μ. Collecting the above arguments, we see that we can bound a graph with no lines generated by the $\mathscr{R}$ action by

$$
\begin{aligned}
&E_{\mathscr{G}} M^m C^m \prod_{v \geqslant v_0} 2^{[n^i_{0v}/2 + 3n^i_{1v}/2 - 2(s_v-1)]h_v} \prod_i 2^{h_i \varepsilon_i} \\
&\quad \leqslant E_{\mathscr{G}} M^m C^m 2^{h[(2m_1+m_2+4m_3-n^e_0)/2 + 3(m_2-n^e_1)/2 - 2(m_3+m_2+m_1-1)+m_1]} \\
&\prod_{v \geqslant v_0} 2^{(h_v - h'_v)[(2m_{1v}+m_{2v}+4m_{3v}-n^e_{0v})/2 + 3(m_{2v}-n^e_{1v})/2 - 2(m_{1v}+m_{2v}+m_{3v})+2+m_{1v}]} \\
&\quad \equiv E_{\mathscr{G}} M^m C^m 2^{-h(n^e_0/2 + 3n^e_1/2 - 2)} \prod_{v \geqslant v_0} 2^{-(h_v - h'_v)[(n^e_{0v} + 3n^e_{0v})/2 - 2]} \qquad (10.9)
\end{aligned}
$$

The formula (10.9) holds for a general graph in which the $\mathscr{R}$ operation is trivial: note that the factor in front of the product in the last line of (10.9) is $2^{-h\delta_P}$ if δ_P is the dimension of P introduced in (7.3). Similarly, the coefficients of $(h_v - h_{v'})$ in the last product are the dimensions of the operators P_v that would be described by the graphs G_v. For the graph under analysis,

with four external lines of type ψ, the factor $2^{-h\delta_P}$ in front of the last product has value 1 and the rhs of (10.9) provides an estimate of the size of the graph, according to its definition in Section 9 (item II).

Remembering that we are still considering graphs contributing to the beta function for the running form factors λ_h, we consider now the second case: the $\mathscr{R}$ actions give rise to graphs in which some of the half lines coming out of some of the clusters $v \in \vartheta$ have changed their original meaning, giving rise, in the given graph G, to lines of type D or S. We put, however, the restriction that the external lines, before the application of the $\mathscr{L}$ operation associated with the first vertex v_0 of the tree ϑ, are local lines of $\psi^{\pm}$ type.

We proceed as in the previous case, when no lines were affected by the $\mathscr{R}$ operation. Let μ_0 be a spanning tree of hard lines, as introduced before (10.6). This time, after completing the relabelings of the graph expressing the results of the action of the $\mathscr{R}$ operations on the clusters of the tree ϑ, we see that some of the half lines may change meaning. As a consequence we associate with μ_0 a new spanning tree μ obtained from μ_0 by considering each line of μ_0 and imagining that it connects the two points out of which emerge the two half lines composing it after the relabeling. The tree μ is again a spanning tree, as can be checked by taking into account the relabeling procedure described in Section 9.

Let λ be an inner half line with frequency index $\bar{h}$. We suppose that the half line λ is one of the half lines affected by the $\mathscr{R}$ operations, so that it represents a nonlocal field Φ among those in (7.1). For the sake of definiteness, we take Φ to be a field of two space-time indices (i.e., $\Phi \neq S^3$): let x, y be the two space-time labels of Φ. If the other half line merging with λ to form a inner line l is supposed, again for definiteness, not to be affected by the $\mathscr{R}$ operation and emerges from the space-time point w, the propagator of the full inner line l will be

$$2^{\eta_\lambda \bar{h}}[g(2^{\bar{h}}w - 2^{\bar{h}}y) - g(2^{\bar{h}}w - 2^{\bar{h}}x)]^{*z_\lambda} \tag{10.10}$$

where the $*z$ symbol means that the difference in (10.10) has to be taken to order z_λ if z_λ is the order of the zero in the field D or S that we are considering; here by *difference to order* z, denoted $[f(y)-f(x)]^{*z}$ of a function $f(x)$ between the point y and x, we mean that the expression is the remainder to order z of the Taylor expansion of $f(y)$ around the point x [see (10.15) below]. To avoid cumbersome notations, the g in (10.10) is not given all the indices and labels that one may think it deserves.

If the half line affected by $\mathscr{R}$ was contracted with another nonlocal half line, the argument would not change, except that one would have to take a difference of an appropriate order also in the variables of the other half line.

Then we can write concisely

$$[g(2^{\bar h}(w-y))-g(2^{\bar h}(w-x))]^{*z_\lambda}=g(2^{\bar h}(w-x),\,2^{\bar h}(w-y))[2^{\bar h}(y-x)]^{z_\lambda} \tag{10.11}$$

defining the tensor g with two arguments by the rhs. In the rhs of (10.11) the terms written symbolically as powers of z_λ represent tensors with z_λ indices, which we regard as contracted with g.

Let $\tilde x_1=y$ and $\tilde x_2,\dots,\tilde x_{p+1}=x$ be the spanning tree path in μ connecting x to y, so that we can write $y-x$ as a *chain sum*: $(y-x)=\sum(\tilde x_i-\tilde x_{i+1})$.

Suppose that (10.11) is substituted in (10.4) and each $y-x$ is substituted by its chain sum and each $(y-x)^{z_\lambda}$ is expanded in a product of single increments corresponding to the lines of the path in μ connecting x to y. This procedure generates various terms like (10.4) in each of which a propagator of a hard line l of frequency h_v is multiplied by a factor of the form

$$\prod_{h<h_v}{}^{*}\,(2^h x_l)^{z_h} \tag{10.12}$$

where x_l is the vector joining the extremes of the line and with the $*$ recalling that the product extends over a (possibly sparse or empty) subset of the set of the scales h less than h_v. The precise subset of values is determined by the graph.

The integer z_h is bounded by the sum of the orders of zero of the lines generating a single contribution in (10.12): this is bounded by 3.

In fact, if a term $2^h x_l$ is present in (10.12), it must come from the propagator of a line containing a nonlocal half line emerging from a cluster v' of frequency $h'>h$, with its space-time indices inside some inner cluster $\bar v$ of frequency $\bar h\geqslant h'$. With this remark in mind, an inspection of (7.5), (7.6), and (7.8) shows that $z_h\leqslant 3$.

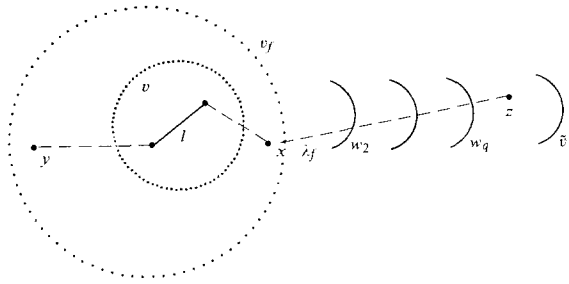
Given a hard line l of frequency h_v for which the product (10.12) is not trivial, let h be the frequency of a half line whose change of meaning due to the $\mathscr{R}$ action on the larger cluster $v_f\leqslant v$ causes the presence of the factor $2^h x_l$ in (10.12), and let λ_f be the half line which has changed meaning; we call $\tilde v$ the cluster to which it belongs ($h\equiv h_{\tilde v}$). Here we recommend drawing a schematic picture of the clusters and of the lines involved [see (10.14)]. We call $p=h_v-h_{v_f}$ and write

$$2^h x_l=2^{h-h_v}(2^{h_v}x_l)=2^{(h-h_{v_f})}2^{-p}(2^{h_v}x_l) \tag{10.13}$$

so we see that we can extract a factor $2^{z_{lf}(h-h_{v_f})}$ from each line contributing to the products in (10.12).

Let $w_1 = v_f > w_2 > \cdots > w_q$ be the set of clusters which the line λ_f successively *crosses*; then

$$h_{v_f} - h \equiv (h_{w_1} - h_{w_2}) + \cdots + (h_{w_q} - h_{\bar{v}})$$



(10.14)

Suppose that there are several lines which have changed meaning because of the $\mathscr{R}$ action and which cross the cluster w and have frequency lower than or equal to that of the immediately larger cluster w'. Let z_w be the total dimensional gain (see Section 7), of the Wick monomial that is represented by the subgraph G_w corresponding to w; then one can collect out of the powers $2^{(h-h_{v_f})}$ a factor $2^{-z_w(h_w - h_{w'})}$; hence the product of the factors (10.12) *over all* v's can be rewritten

$$\left\{\left(\prod_v 2^{-(h_v - h_{v'})z_v}\right)\prod_v\left[\prod_{l \in v}\prod_{(z,p) \in S_0(v,l)}[2^{-p}(2^{h_v}x_l)]^z\right]\right\} \tag{10.15}$$

where $S_0(v, l)$ denotes a set of pairs of integers (z, p), such that $p \geqslant 0$, $z = 1$, 2, or 3, and there is at most one pair for each p; $S^0(v, l)$ depends on special graphs and, given the graph (as in our case), on the special terms singled out in the expansion giving rise to (10.12).

We take out of the graph value the first factor in (10.15) and consider what remains. We want to show that all the factors of the other products in (10.15) containing the same x_l can be put together with the propagator of the line l leading to an evaluation of the same graph with more complicated propagators, which, however, are easy to bound.

We first deal with a better representation of the two-argument propagators g in the rhs of (10.11). The treatment of four-argument propagators is clearly similar, but for simplicity of exposition of an already intricate discussion, we are supposing (as stressed several times above) that there are no four-argument propagators, i.e., no line results in a contraction of $\mathscr{R}$-affected half lines, or, in still other words, no contraction DD or DS or SS is present in the considered graph.

Rather than carrying around propagators with more than two space-time indices, we write (10.11) via the interpolation formula

$$[f(y)-f(x)]^{*z}=\int_0^1 \frac{(1-t)^{z-1}}{(z-1)!}\frac{\partial^z f}{\partial x^z}[x+t(y-x)](y-x)^z\,dt \qquad (10.16)$$

so that the function g with two arguments in (10.11) can be replaced, by using (10.15), by a new function g with one argument considered at an interpolated point.

Let l be a (hard) line in the spanning tree μ of scale h_v and let x_l be the vector joining the two extreme points of l. Collecting all terms containing the same x_l, we see that (10.15) will allow us to consider that the propagator of l will have the form

$$g(2^{h_v}(x_l+t_l r_l))\prod_{(z,p)\in S_0(v,l)}[2^{-p}(2^{h_v}x_l)]^z \qquad (10.17)$$

where t_l is an interpolation parameter in $[0,1]$, and r_l is the vector joining the endpoint of l, from which the nonlocal half of l emerges, and the other space-time point of the nonlocal half line; of course l may be composed of two local lines: in this case we have no $t_l r_l$ term in (10.17) and we take r_l equal to zero.

More precisely, our value of the graph G on the tree ϑ can be computed by pretending that each hard line has a propagator given by (10.17) and then:

1. Integrating over the interpolation parameters with respect to a suitable measure on the t's, giving measure $\leqslant 1$ to the whole space of the interpolating parameters [and such a measure has a density which, by (10.15), is a product of powers of t-variables].
2. Summing over all the possible choices of the sets S_0.
3. Multiplying the result by the first factor in (10.15).

If $r_l\neq 0$, we introduce $\xi_l=2^{h_v}(x_l+t_l r_l)$ and rewrite (10.16) as

$$g_\lambda(\xi_l)\prod_{(z,p)\in S_0(v,l)}[2^{-p}(\xi_l-2^{h_v}t_l r_l)]^z \qquad (10.18)$$

The monomials $2^{-p}(\xi_l-2^{h_v}t_l r_l)^z$ can be developed and one gets many terms, for each of which one can pretend that the propagator is

$$g_l(\xi_l)\left[\prod_{(z_0,p_0)\in S_0}(2^{-p_0}\xi_l)^{z_0}\right]\left[\prod_{(z',p')\in S'}(2^{-p'}2^{h_v}r_l)^{z'}\right] \qquad (10.19)$$

provided that, after the evaluation of the graph is completed, one integrates over the interpolation variables and sums over the possible choices of the sets S_0 and S'.

We now remark that $r_l 2^{h_v}$ can be again rewritten as

$$2^{-(h_{\bar{v}}-h_v)} \sum_x 2^{h_{\bar{v}}} x$$

where x is the vector in the spanning tree μ in the path leading from one to the other of the two space-time indices associated with the nonlocal half of l; $\bar{v}$ is the cluster out of which l emerges at the scale $h_{\bar{v}}$ on which the action of $\mathscr{R}$ changes its meaning. We see that $h_{\bar{v}} - h_v \geqslant 1$, and that $h_{\bar{v}} \leqslant h_{v'}$, if $h_{v'}$ is the frequency of the line associated with x.

Hence, developing the monomials

$$(2^{-p'} 2^{h_v} r_l)^{z'} \equiv \left(2^{-p'} \sum_x 2^{h_{v'}} x \cdot 2^{(h_v - h_{v'})} \right)^{z'}$$

we find that the computation of G can be made by integrating over the interpolating variables t and summing over suitable S_0, S_1 the value of the graph in which l has a propagator:

$$g(\xi_l) \left[\prod_{(z_0, p_0) \in S_0} (2^{-p_0} \xi_l)^{z_0} \right] \left[\prod_{(z_1, p_1) \in S_1} (2^{-p_1 - 1} 2^{h_v} x_l)^{z_1} \right] \tag{10.20}$$

with S_0, S_1 defined as above; this means that the product over l of the (10.18) equals the product over l of (10.19) if S_0, S_1 are conveniently chosen (for each l).

We can now repeat the argument until we can reach a situation in which the x_l are replaced by the corresponding ξ_l, which happens sooner or later because the innermost lines of our graph are necessarily built with pairs of local lines.

At the end of the process we shall be able to compute the graph value by taking the hard lines l to have propagators

$$g^{(h_v)}[2^{h_v}(x_l + t_l r_l)] \prod_{j \in J} \left\{ \prod_{(z, p) \in S_{j,l}} 2^{-p-j} [2^{h_v}(x_l + t_l r_l)]^z \right\} \tag{10.21}$$

where the above products run over a finite set of indices, depending, however, on the particular case considered (i.e., graph, tree, selected terms, etc).

The soft line propagators have the form $g^{(<h_v)}[2^{h_v}(x_l + t_l r_l)]$ with no extra factors. Of course l may again be composed of two local lines: in this case we take r_l equal to zero as above. And after the evaluation of the

graph with the above propagators one has to integrate over the interpolation variables (with respect to a measure with total variation bounded by 1) and then one sums over a convenient set of choices of the sets $S_{j,l}$: since clearly *very little control* can be hoped for the choices of such sets, we shall bound the result by considering them completely arbitrary.

The main property of (10.21) is that, if summed over an arbitrary set of choices of S_0, S_1,..., it still is a function of $x \equiv 2^{h_v}(x_l + t_l r_l)$ which is C^∞ and decays fast at ∞, with all its derivatives, uniformly in h_v and uniformly in the choices of the sets of S_j to be used in the sum.

The reason is quite simple and we just check here the bounds for the function itself (for its derivatives one just differentiates and repeats the argument). By taking the absolute values in (10.20) and summing over all sets S_0, S_1,..., we find a bound

$$|g^{\varepsilon,(h_v)}(x)| \prod_{p=0}^{\infty} \prod_{j=0}^{\infty} \left[1 + \sum_{z=1}^{3} |(2^{-p-j}x)\, \tilde{g}^{\varepsilon}(2^{-p-j}x)|^z \right] \tag{10.22}$$

where we only use that it is possible to write

$$g^{(h)}(x, \boldsymbol{\omega}) = g^{(h),\varepsilon}(x, \boldsymbol{\omega}) \prod_{j=0}^{\infty} \prod_{p=0}^{\infty} \tilde{g}^{\varepsilon}(2^{-j-p}x) \tag{10.23}$$

with $g^{(h),\varepsilon}(x)$, $\tilde{g}^{\varepsilon}$ still analytic in a strip around the real x axis and uniformly bounded and decaying exponentially fast at infinity together with any pre-fixed number of derivatives [see (A.5) and the holomorphy argument following it]. To see what is going on, let us consider the model in which the g_h are supposed to coincide with their asymptotic form (A.7) and (A.8): we see in this case that the product of the $\tilde{g}^{\varepsilon}(x)$ in (10.23) could be taken: $\exp[-\varepsilon \sum_{j,p=0}^{\infty} (2^{-j-p}x)^2]$.

We can evaluate the graph value using again the Fourier transform and we see that, if we call k the momentum variable associated with the line x, then we have extra momentum tk entering or leaving the vertices at the extremes of the line defining r, if t is an appropriate interpolation variable.

But we can write r as the sum of the vectors associated with the hard lines in the spanning tree μ connecting its two extremes, so that we can think that the momentum tk is in fact added to all the hard lines in μ connecting the two extremes of r. It is therefore clear that it only affects the construction discussed in the simple case (with no nonlocal lines in it) *by changing the values of some* of the momentum variables k appearing in the hard lines: however, such k's disappear when one performs the integration over the ω's and one is left with the same integral over the loop variables; the conditions under which this is correct are that the propagators in

(10.21) have the same properties used in the easy case, and that the sum over the subsets of the sets S_0, S_1,... can be performed. The two conditions are met because of the bound (10.22) and the analogous bounds on the derivatives.

This shows that we get, in this case, the same bound found in the absence of the $\mathscr{R}$ generated lines with different constants and an extra factor $\prod_v 2^{z_v(h_{v'}-h_v)}$, where z_v is the total dimensional gain of the vertex v and v' is the vertex preceding v in the tree ϑ.

Hence the size of the relevant graph leading to (10.4), i.e., the integral in (10.4), is bounded by the rhs of the following expression:

$$E_\vartheta M^m C^m \left(\prod_{v \geqslant v_0} 2^{h_v[n^i_{0v}/2 + 3n^i_{1v}/2 - 2(s_v - 1)]}\right)\left(\prod_{v \geqslant v_0} 2^{-(h_v - h_{v'})\, z_v}\right)\left(\prod 2^{h_i \varepsilon_i}\right)$$

$$\leqslant E_\vartheta M^m C^m 2^{-h\delta_P} \prod_{v \geqslant v_0} 2^{-(h_v - h_{v'})\, \rho_v} \tag{10.24}$$

by the same argument given in (10.9); here $\rho_v = (n^e_{0v} + 3n^e_{1v})/2 - 2 + z_v$, where z_v is the dimensional gain in the half lines emerging from the subgraph G_v, δ_P is the dimension [which is in fact 0 in our case; see comment following (10.9)], and M is introduced after (10.8).

The bounds on the size C_0 of the graphs can be extended to the estimate of the bounds of the sizes C_N, $N > 0$: the result is a simple consequence of the above analysis. The C_N bounds have the same form as the C_0 bounds with new constants C, D. The obvious modifications of the above analysis imply that the numbers z, bounded by 3 for C_0, are now bounded by $3 + 2N$, and the final constants D, C depend on N. The fact that the external momenta are $k_1,..., k_n$ rather than zero makes no difference, as the value of the external momenta was never used in the C_0 bounds. This can be interpreted as saying that the effective potential kernels have a short range, uniformly in h.

The above analysis can be immediately extended to cover the bounds on the C_N size of irrelevant graphs with any number of external lines (local or not): the final bound looks exactly the same as (10.24) because of the extra factors present in this case in the definition of the size (see Section 9, item II); the definition in (9.3) has just been set up, in the general case, to make the latter statement true as a consequence of the above analysis, thereby providing a convenient way of summarizing the results of this section.

The bounds for the relevant graphs with two external lines which, before the action of the $\mathscr{L}$ operation relative to the first nontrivial vertex v_0 of the tree ϑ represent graphs with two external lines of type $\psi^+\psi^-$, are performed in a identical fashion, again leading to a bound like (10.24). The

constants in all the above bounds can be taken to be the same, and h independent.

Therefore the cases that have not yet been treated are those corresponding to graphs contributing to the relevant part, which, before the localization $\mathscr{L}$ relative to the first tree vertex, have (a) four external lines, one of which is a D^+ line, while the three others are $\psi^\pm$ lines; (b) two external lines, one of which is a D^+ line.

Looking at (7.6) and (7.8), we see that the above two cases are precisely the ones for which the application of the $\mathscr{L}$ operator produces differences of local terms evaluated at different points, say x_1 and x_2.

We can proceed to bound their contribution to the beta functional in the way followed above, simply estimating each term in the difference of local terms without trying to exploit possible cancellations between them: i.e., we give up using the fact that when $x_1 = x_2$ such terms vanish.

Then a bound like (10.24) still holds: however, the numbers z_v associated with the chain of clusters v containing the two points x_1, x_2 on which the $\mathscr{L}$ operator generates the local terms are smaller by one unit compared to those that would be *a priori* necessary to make the coefficient $\rho_v = (n^e_{0v} + 3n^e_{1v})/2 - 2 + z_v$ of $h_v - h_{v'}$ greater than or equal to 1. Hence, some of the ρ_v corresponding to vertices v containing x_1, x_2 may be zero and therefore they would ultimately produce logarithmic divergences in h when the summation over the h_v is performed to compute the final expression for the contributions to the beta functional.

This mild *lack of uniformity* in h of the beta functional is the best we can do without taking into account special new features. Since the *uniformization* of the bounds will be done by using different mechanisms in the $d = 3$ and in the $d = 1$ cases, we discuss it in the next section.

11. UNIFORMIZATION OF THE BOUNDS ON THE BETA FUNCTIONAL. LOOP IMPROVEMENTS

To complete the perturbative theory of the beta functional, we have to improve the bounds of Section 10 in the two remaining cases (a), (b) described at the end of Section 10. Also in this section we shall consider explicitly only the case in which there are, in G, only relevant vertices.

We start by considering the case of graphs with two external lines, i.e., contributions to the beta functional for ν, α, ζ. In this case it suffices to remark that the constants ν_h, α_h, ζ_h are independent of the values $\boldsymbol{\omega}$, $\boldsymbol{\omega}'$ of the quasimomenta of the fields in the Wick monomials. Therefore we can evaluate the contribution to the variation of ν, α, ζ by choosing to compute them when $\boldsymbol{\omega} = \boldsymbol{\omega}'$.

Consider first the contribution coming from the application of $\mathscr{L}$ to a graph which, before applying $\mathscr{L}$, produced a Wick monomial $D^{+}_{x_1 x_2 \omega} \psi^{-}_{x_3 \omega'}$. Using (7.8), one sees that the $\mathscr{L}$ action produces an operator, for $\omega = \omega'$,

$$\int v(x_1 - x_2, x_1 - x_3, \omega)\, e^{ip_F(x_1 - x_3)\omega}\, dx_1\, dx_2\, dx_3\, [:\psi^{+}_{x_1\omega}\psi^{-}_{x_1\omega}: - :\psi^{+}_{x_2\omega}\psi^{-}_{x_2\omega}:$$

$$+ (x_3 - x_1):\psi^{+}_{x_1\omega}\mathscr{D}\psi^{-}_{x_1\omega}: - (x_3 - x_2):\psi^{-}_{x_2\omega}\mathscr{D}\psi^{-}_{x_2\omega}:] \tag{11.1}$$

where v is a suitable kernel. The (11.1) contributes both to ν and to ζ, α. The first contribution arises from the terms $\psi^{+}\psi^{-}$ and is

$$\int dx_2\, dx_3\, v(x_1 - x_2, x_1 - x_3, \omega)\, e^{ip_F\omega(x_1 - x_3)}$$

$$- \int dx_1\, dx_3\, v(x_1 - x_2, x_1 - x_3, \omega)\, e^{ip_F\omega(x_1 - x_3)} = 0 \tag{11.2}$$

The contribution to α, ζ is not zero because of the $(x_3 - x_1)$, $(x_3 - x_2)$ factors, which break the symmetry between 1 and 2 in the fields: by adding and subtracting suitable terms we see that the contribution reduces to

$$\int dx_1\, dx_3\, v(x_2 - x_1, x_2 - x_3, \omega)\, e^{ip_F\omega(x_2 - x_3)}(x_2 - x_1):\psi^{+}_{x_2\omega}\mathscr{D}\psi^{-}_{x_2\omega}: \tag{11.3}$$

In this case we do not get zero; however, we see that we can explicitly exhibit the factor $(x_2 - x_1)$. Repeating the estimates of Section 10, we see that we can bound the contribution using (10.24), in which we take $n^e_0 = 1$, $n^e_1 = 1$ with $\rho_v \geqslant 1$, getting a uniform bound as $h \to -\infty$.

For the same reason all the other contributions to α, ζ, ν that one might expect from (7.8) and not taken into account in Section 10 vanish (as the 1, 2 indices appear symmetrically in the fields).

Consider now the case of graphs with four external lines, i.e., consider the contributions to the beta functional for λ_h. We see that the above cancellations occur only if the ω_i are in a *Cooper pair* configuration: $\omega_1 = -\omega_2$; $\omega_3 = -\omega_4$. However, we cannot infer from the gauge symmetry that it is enough to consider λ_h on such configurations to know it on all the others.

There is one remarkable exception to this situation: namely if $d = 1$ and the spin is zero, it turns out that the Cooper pair configurations of quasimomenta do determine the λ_h: simply because they are the only configurations for which $\psi^{+}_{x\omega_1}\psi^{+}_{x\omega_2}\psi^{-}_{x\omega_3}\psi^{-}_{x\omega_4}$ does not vanish identically (by the exclusion principle). Hence if $d = 1$ the contributions to the beta function

due to the cases (a) and (b) considered above can be taken to vanish and we have uniform bounds even in the contributions to λ_h.

The above remark suggests introducing the notion of *Cooper pairs* of quasiparticles with quasimomenta $\pm\omega$. We shall say that a local fourth-degree term like

$$I = \int \lambda(\omega_1, \omega_2, \omega_3, \omega_4)\, e^{ip_F(\omega_1+\omega_2-\omega_3-\omega_4)x} \times :\psi^+_{x\omega_1}\psi^+_{x\omega_2}\psi^-_{x\omega_3}\psi^-_{x\omega_4}: \prod d\omega_j \tag{11.4}$$

contains a component $\mathscr{P}_C I$ of interaction between Cooper pairs defined by setting

$$\lambda(\omega_1;\omega_3) = \lambda(\omega_1, -\omega_1, \omega_3, -\omega_3)$$
$$P_C\lambda(\omega_1,\omega_2,\omega_3,\omega_4) = 4^{-1}[\lambda(\omega_1;\omega_3)-\lambda(\omega_1;\omega_4)-\lambda(\omega_2;\omega_3)+\lambda(\omega_2;\omega_4)] \tag{11.5}$$

and

$$\mathscr{P}_C I = \int P_C\lambda(\omega_1,\omega_2,\omega_3,\omega_4)\, e^{ip_F(\omega_1+\omega_2-\omega_3-\omega_4)x} \times :\psi^+_{x\omega_1}\psi^+_{x\omega_2}\psi^-_{x\omega_3}\psi^-_{x\omega_4}: \prod d\omega_j \tag{11.6}$$

Calling $\mathscr{L}_C = \mathscr{P}_C\mathscr{L}$ and $\mathscr{R}_C = 1-\mathscr{L}_C = \mathscr{R}+(1-\mathscr{P}_C)\mathscr{L}$, we can try to repeat the analysis of Sections 6–10 using $\mathscr{R}_C$, $\mathscr{L}_C$ rather than $\mathscr{R}$, $\mathscr{L}$.

The $\mathscr{L}_C$ operation is quite complicated (in the fourth-degree part) if expressed as an operator on the fields:

$$\begin{aligned}
\mathscr{L}_C :&\psi^+_{x_1\omega_1}\psi^+_{x_2\omega_2}\psi^-_{x\omega_3}\psi^-_{x\omega_4}: \\
&= \delta(\omega_1+\omega_2)\,\delta(\omega_3+\omega_4)\Big[\int d\omega_2'\, d\omega_4' \\
&\quad \times :(\psi^+_{x,\omega_1}-\psi^+_{x,-\omega_1})\,\psi^+_{x,\omega_2'}(\psi^-_{x,\omega_3}-\psi^-_{x,-\omega_3})\,\psi^-_{x\omega_4'}: \\
&\quad \times e^{ip_F(\omega_1+\omega_2'-\omega_3-\omega_4')x}\Big]_{AS} \\
\mathscr{L}_C :&\psi^+_{x_1\omega_1}\psi^-_{x_2\omega_2}: = :\psi^+_{x_1\omega_1}S^1_{x_2x_1\omega_2}:
\end{aligned} \tag{11.7}$$

It is easy to check that this time $\mathscr{L}_C$ vanishes on any fourth-degree monomial containing at least one nonlocal field or a $\mathscr{D}\psi$ field: this means that the problem mentioned above, due to the fact that the latter property does not hold for the $\mathscr{L}$ operation [by (7.6)], will not arise.

But, having changed the rules of the game, we must reexamine Section 10 to check if it still would work with the new definition: it is not difficult to realize that it does not, unless we modify the assumptions on $|\lambda_h|$ strengthening it. We shall come back to this approach in Section 14, where we discuss our conjectures about the flow generated by the beta functional. Here we adopt a different viewpoint.

We consider the action of $\mathscr{R}$, $\mathscr{L}$ on Wick monomials which appear in the graphical representation of the effective potential. The linearity of $\mathscr{L}$ has been used so far only on the second-degree terms; for all other purposes we could as well use a different definition of $\mathscr{L}$. We could define it as a nonnecessarily linear operation acting on each of the operators that are produced via the graphical rules of Section 9. See the concluding remarks to Section 7.

In this way we lose gauge invariance: so far the use of gauge invariance has been just a way of getting rid of unwanted terms in the second-degree part of the effective potential. Later we shall use it again to analyze the pair Schwinger functions: but we shall need it only in the second-degree terms of the effective potential. Hence we can modify the definition of $\mathscr{L}$ when it acts on the fourth-degree terms of the effective potential, keeping it as before when it acts on the second-degree part. The point is that the $\mathscr{L}$ can be easily modified so that the above problems on the beta functional disappear.

The choice will be simply to take $\mathscr{L}$ to be zero in the cases (7.6) and to be the same as before in all other cases: consequently $\mathscr{L}$ will vanish on all fourth-degree monomials which involve nonlocal field or $\mathscr{D}\psi$ fields.

It is clear that this *slight* modification does not introduce any new problem in Section 10 and all the estimates there remain unchanged. The reason is that the terms containing a D^+ field, like the ones in (7.6), already have the correct power counting and the subtraction (7.6) is only necessary if one wishes to keep the linearity of $\mathscr{L}$ on the fourth-degree terms. We do not lose the linearity (hence the gauge invariance) on the second-degree terms.

With the above modification of $\mathscr{L}$, $\mathscr{R}$ on the fourth-degree operators we have completed the proof of the uniformity of the beta functional bounds in terms of the frequency parameter h, i.e., of the scale of the root vertex of the tree.

We now proceed to try to identify classes of graphs whose contribution to the beta functional can be shown to be bounded by

$$2^{\varepsilon h} C_\varepsilon^{m-1} D_\varepsilon \prod_v \frac{2^{(1-\varepsilon^1)\rho_v(h_v - h'_v)}}{s_v!} \tag{11.8}$$

for some ε, $\varepsilon^1 > 0$; i.e., we want to see if we can make use of the fact that

$\rho_v \geqslant 1$ and of the fact that in the theory of the beta functional one only needs, as is well known,[18] that $\rho_v \geqslant \varepsilon$.

A large number of graphs can be shown to obey the improved bound (11.8). This is due to some basic inequalities or remarkable cancellations which we call *loop improvements*, as they manifest in a nontrivial way in graphs with loops. They play a role not only in the theory of the beta functional (which is greatly simplified), but in the theory of the Schwinger functions as well (see Section 12).

Consider first the case $d=3$. In this case one makes essential use of the following *three-quasimomentum inequality*, which follows by simple phase space considerations [see Appendix B for the general technique, and (B14) for a check]:

$$\int \delta_{h'}(\omega_1+\omega_2-\omega_3+\kappa)\, d\omega_1\, d\omega_2\, d\omega_3 \leqslant C_1 2^{h'}, \qquad \forall \kappa \tag{11.9}$$

We stress that this holds only if $d>1$; here $\delta_h(\mathbf{k}) = 2^{-2h} g(2^{-h}\mathbf{k})$, $k \in R^d$, with g a short-range function.

Recall that a basic estimate in Section 10 was the bound of

$$\int \delta_{h'}(\omega+\kappa')\, d\omega \leqslant C_1, \qquad \forall \kappa' \tag{11.10}$$

where ω was the quasimomentum of a hard line of frequency h' associated with the approximate delta functions $\delta_{h'}$ and κ' a suitable linear combination of momenta and quasimomenta; see, for instance, the comments after (10.8).

Then we see that we can gain a factor $2^{h'}$ by using (11.9) if we can show that κ' in (11.10) has the form $\omega_1+\omega_2+\kappa$ with ω_1, ω_2 being quasimomenta of internal lines which are not in the spanning tree. We might even be able to gain similar factors several times by applying the remark to different hard lines of the spanning tree: provided, however, that we are able to identify distinct quasimomenta ω_1, ω_2 for each different hard line of the spanning tree for which we want to obtain a gain.

We use the above inequality to show that in bounding some graphs we can improve the bound of Section 10 [i.e., (10.24) in dimensionless form, without the factor $2^{-h\delta\rho}$]

$$M^m C^{m-1} D \prod_v \frac{2^{-\rho_v(h_v-h'_v)}}{s_v!} \tag{11.11}$$

by a factor 2^{h_0}, where $h_0 = h_v$ for some v. Clearly this immediately implies the validity, for the given graph, of the bound (11.8).

We begin by examining the case of the contributions to the effective potential from graphs (relevant or not) with two external lines. We fix a

tree ϑ and a graph G with two external lines. Let v_0 be the first nontrivial vertex and let $v_1,...,v_s$ be the tree vertices immediately following v_0. The frequency of v_0 is $h+1$. We do not consider local contributions arising from the terms which we have shown to cancel when the external quasimomenta are equal.

There are two possibilities to examine. The first is that one of the internal lines of frequency h connecting the cluster v_1 to the others is essential for the connectedness of the graph, i.e., cutting it, the graph becomes disconnected: this case is the easiest when it cannot be reduced to the next one and we therefore postpone its treatment.

The second possibility is that none of the inner lines of frequency h connecting v_1 to the other clusters is essential for the connectedness of the graph. This means that if one shrinks to a point all the subgraphs inside each of the clusters v_i, the graph $\tilde{G}$ that one obtains contains at least one loop.

Suppose that in $\tilde{G}$ there are at least two lines l, l', with quasimomenta ω, ω', which are not in the spanning tree. Then it is clear that one of the approximate delta functions associated with the hard lines of the spanning tree must have the form $\delta_h(\pm\omega\pm\omega'\pm\omega_1+\kappa)$ with ω_1 being the line quasimomentum, for a suitable arrangement of the signs. Hence we can apply (11.9) at least once and we improve the final bound by a factor 2^h. Hence we are left with the case of $\tilde{G}$ with just one loop. It is easy to see that we can repeat the argument even in this case as long as the graph G has at least two loops, even if one of them is inside one of the v_j. The conservation of total momentum at each vertex implies, in fact, that two at least of the quasimomenta of the lines inner to the v_j but not on the spanning tree must appear also in the momenta of a hard line on some scale h_0; hence one can easily check that (11.9) can be applied at least for one such line and improves (11.11).

Therefore the only case (if the second possibility foreseen at the beginning is verified) in which we have not proved the presence of an extra power of 2^{h_0} in the bound (11.11) is when the graph G has only one loop: clearly this can happen when it contains only one vertex with four lines and all the others are two-line vertices. This means that only the contributions to the effective potential of degree two in the fields or to the beta functional for α, ζ, ν which are linear in λ and at least linear in α, ζ, ν may not have an extra factor 2^{h_0} in front of their bound (11.11). The possibility of treating this case is based on a remarkable cancellation which is a generalization of the one exhibited in a very special case in the calculation in Appendix B.

We first replace all propagators with their leading expressions; we also replace the operators $\mathscr{D}_\omega$ by their leading expression $\beta\omega\partial$. All the correc-

tion terms contain at least one $2^{h'}$ for some h' and hence by the remark following (11.11) we see that we get the bound (11.8) on this part of the estimate.

The second operation is to *undo* some of the $\mathscr{R}$ concerning the internal clusters according to the rule explained below. Then, if the first cluster bears an $\mathscr{R}$ label, we replace the nonlocal external field in the expression of the operator associated with the graph by the appropriate tensor describing the structure at contact of the zero of the nonlocal fields [see (7.3)]; then we multiply by the appropriate 2^{-h} and by the factors necessary to take the Fourier transforms in (9.3). The resulting expression, integrated over the space-time indices distinct from the localization point, provides the estimate for the C_0 size of the operator. If the first cluster bears an $\mathscr{L}$ label, we simply write the corresponding expression for the C_0 size.

For simplicity we examine the case of a contribution to the beta functional v term (i.e., a graph with an $\mathscr{L}$ operation selecting a $\psi^+\psi^-$ term).

Since our graph is topologically very simple, we can represent the various terms arising in undoing the $\mathscr{R}$ operations [$\leqslant 6^n$, by (7.5)–(7.8), if n is the number of two-line graph elements] rather naturally in terms of new graphs constructed as follows.

We draw a one-loop graph with $s < n+1$ vertices each of which represents one of the s maximal clusters of the tree. If v is one of the clusters with two external lines, we make a choice and either we leave the graph as it is, deleting the $\mathscr{R}$ label, or we draw a *hanging* chain emerging from v with s_v vertices representing the s_v clusters inside v, the first being v itself and the others labeled from 2 to s_v. Inside each of the hanging chain vertices we imagine drawn the corresponding subgraph.

Similarly, if the cluster v has four external lines, we see from a simple analysis that one of the four has to be a D_{xy} line which, once *undone* in its two ψ fields, can be represented naturally by two graphs the first of which is the starting one without the $\mathscr{R}$ label on v. The other is obtained from it by imagining the external line corresponding to the first point x of the D_{xy} field as emerging from the second point y of the D field, while the chain of the external line maximal clusters, linking x to the maximal cluster inside v with four external lines, will again be replaced by a *hanging* chain of vertices. Inside each of them we imagine drawn the corresponding chain graphs.

We undo the $\mathscr{R}$ operations involving the maximal clusters; then we undo the $\mathscr{R}$ operations concerning the (only) cluster of the next generation with four external lines and so on until there is no longer any cluster with four external lines which is not a point.

The value of the factor contributed by each hanging chain is easily

estimated using the results of Section 10 and depends on the term that is selected when undoing the $\mathscr{R}$ operation which generates the hanging chain. If the $\mathscr{R}$ operation concerned a two-external-line cluster, the local term that appears when undoing it (which generates, in the graphical interpretation, the hanging chains in the graph) corresponds to an operator $n{:}\psi^+_{x\omega}\psi^-_{x\omega}{:}$ (which has dimension $\delta = -1$) or to an operator $b{:}\psi^+_{x\omega}\mathscr{D}\psi^-_{x\omega}{:}$ (which has dimension $\delta = 0$). If $\mathscr{R}$ concerns a four-external-line cluster, the local term corresponds to an operator $l{:}\psi^+_{x\omega_1}\psi^+_{x\omega_2}\psi^-_{x\omega_3}\psi^-_{x\omega_4}{:}$ (which has dimension $\delta = 0$).

It is clear that the analysis leading to (10.24) implies that the coefficients n, l, b can be bounded by

$$M^{m_v}C^{m_v}2^{-\delta h}\left[\prod_{w\subset v}\frac{2^{-(h_w-h_{w'})\rho_w}}{s_w!}\right]2^{z_v(h_v-h)} \tag{11.12}$$

using the notation of Section 10, (10.24), with the above values for δ. The (11.12) simply takes into account the *missing* factor represented by the last term.

Hence we see that iterating the above arguments to undo the inner $\mathscr{R}$ operations (when prescribed), we are reduced to the main problem of estimating a one-loop graph, computed on a tree with no $\mathscr{R}$ operators acting on the inner vertices, *with the proviso* that we have to multiply the result by a product of $2^{z_v(h_v-h_{v'})}$ over an unspecified set of frequency jumps.

To estimate a one-loop graph with no $\mathscr{R}$ operations acting on the inner clusters and two-line vertices, we essentially compute the integral explicitly. We want to show that the C_0 (and C_1) sizes of such graph can be bounded by

$$M^nC^n2^hn!^4\prod_v\frac{2^{z_v(h_v-h_{v'})}}{s_v!} \tag{11.13}$$

This, together with (11.12), would show that we have an extra 2^h in the bounds of the one-loop graphs with two external lines, at least if are willing to pay the price of the $n!^4$ and of the frequency jump damping factors, which, in fact, are with the wrong sign and provide no damping at all.

We are, however, interested in resumming all the above terms as we want the estimate of the full expression with the $\mathscr{R}$ operations not undone. For such sum we have, from the previous work, an alternative estimate (11.11), with no factorials and with the correct damping factors. Hence we simply interpolate the two bounds [the number of terms obtained by undoing the $\mathscr{R}$ operation is not very large ($\leqslant 6^n$)], using an interpolation parameter $\varepsilon < 1/4$ for the part bounded by (11.13) and $(1-\varepsilon)$ for the part bounded by (11.11). And we find almost (11.8): we still have the factor $n!$ which should not be there.

But there are very *few*, at most $O(\prod_v s_v!)$, graphs with only one loop and we can still say that (11.8) holds if, to avoid notational problems, we decide that the one-loop graphs have a value which is the previously defined one but divided by $n!$ and, to compensate, each graph is repeated $n!$ times. Of course this is only a notational trick and if one wished, one could continue distinguishing the one-loop graphs from the others.

To prove the above claim (11.13), consider a one-loop graph with $n+1$ vertices divided into a hierarchy of subgraphs according to a given tree ϑ. We suppose that no $\mathscr{L}$ or $\mathscr{R}$ operation is performed on the operators representing the subgraphs. Suppose also, again to simplify the analysis by avoiding trivial cases, that the graph contains no 2-insertions on the external lines (a case that is very easily discussed after the main problem on the loop integral is solved).

We note that the C_0 size of such a graph is obtained as the product of the combinatorial factor $E_\vartheta = \prod_v 1/s_v!$ times:

$$2^{-h} \int \left[\prod_{j=1}^{n+1} 2^{h_j} \Omega^j_{\omega_j} g_{h_j}(x_j - x_{j-1}, \omega_j)\, e^{-ip_F(\mathbf{x}_j - \mathbf{x}_{j-1})\,\omega_j}\, d\omega_j \right] \times \lambda_0 \left[\prod_{j=1}^{n} \mu^j_{\tilde{h}_j} \right] dx_1 \cdots dx_n \tag{11.14}$$

where $\Omega^j_{\omega_j}$ is either the identity or the $\mathscr{D}_t$ operator on the coordinate x_j or the $\omega_j \cdot \partial_{\omega_j}$ operator and, correspondingly, $\mu^j_{\tilde{h}_j}$ is $2^{\tilde{h}_j} \nu_{\tilde{h}_j}$ or $\zeta_{\tilde{h}_j}$ or $\alpha_{\tilde{h}_j}$, where $\tilde{h}_j$ is the scale of the jth vertex. In general $\tilde{h}_j > h_j$, which is the scale of the propagator associated with the line from x_{j-1} to x_j. The vertices are labeled from 0 to $n+1 \equiv 0$ following the arrows of the loop lines.

The (11.14) can be rewritten more explicitly by using the Fourier transform expressions for the propagators (we fix the parameter $\beta = p_F/m = 1$):

$$\int dk \left(\prod d\omega_j\, \alpha_j\, d\alpha_j \right) \tau M^{n+1} 2^{h - h_{n+1}} \prod_{j \in J_v} 2^{\tilde{h}_j - h_j} \times (ik_0 + \boldsymbol{\omega}\mathbf{k})\, e^{-\alpha_{j_0} k^2} \prod_{j \neq j_0} 2^{-2h_j} (ik_{0j} + \omega_j \mathbf{k}_j)\, e^{-\alpha_j k_j^2} (\omega_j \mathbf{k}_j)^{a_j} (i\dot{k}_{0j})^{b_j} \tag{11.15}$$

where a_j, $b_j = 0, 1$; M is a bound on ν, α, z, λ; $\tau \in [-1, 1]$ depends only on the $\boldsymbol{\omega}$-variables; J_v is the set of loop lines entering a vertex with running constant of v type; j_0 is the label of the line l_0 from $x_{j_0 - 1}$ to x_{j_0} which is not in the spanning tree of the graph; the interpolation parameters α_j [see (A9)] are in $[1/4, 1]$ for all $j \neq j_0$, and for $j = j_0$, $\alpha_{j_0} \in [1/4, 1]$ or $\alpha_{j_0} \in [1, \infty]$. The actual values of a_j, b_j depend on the type of the jth vertex ($a_j = 1$ if the vertex is of type α, and $b_j = 1$ if it is of type ζ; the a, b vanish

otherwise). The momenta k_j can be expressed in terms of the quasimomenta $\boldsymbol{\omega}_j$ of the jth line (from x_{j-1} to x_j) via

$$2^{h_j}k_j = 2^h k + \boldsymbol{\omega} - \boldsymbol{\omega}_j \tag{11.16}$$

where k, $\boldsymbol{\omega}$ are the momentum and quasimomentum of the j_0th line.

The α_j with $j \neq j_0$ play no role and will be set equal to 1 to simplify the analysis. Some care has to be devoted to α_{j_0}, which we denote by α.

We set $N = \sum_j 2^{2(h-h_j)} \geqslant 1$ (since there is at least one more line of frequency h, besides l_0). We then express the inner momenta k_j in terms of k using (11.16) and we collect all the k^2 in the exponentials after developing the squares. We also collect in the exponentials all the terms linear in $\mathbf{k}$ and we perform the linear change of coordinates necessary to eliminate from the exponentials the terms linear in $\mathbf{k}$. Using the notation

$$\begin{aligned} &\boldsymbol{\Omega}_j = 2^{-h_j}(\boldsymbol{\omega} - \boldsymbol{\omega}_j), \qquad && \varepsilon_j = 2^{h-h_j} \\ &N = \sum_{p \neq j_0} \varepsilon_p^2, && \mathbf{w} = \sum_p \varepsilon_p \boldsymbol{\Omega}_p \end{aligned} \tag{11.17}$$

we find, after some algebra, that the integral expressing the C_0 size of the considered graph can be written as the combinatorial factor $E_{\mathcal{G}} = \prod_v 1/s_v!$ times:

$$\begin{aligned} M^{n+1} 2^{h-h_{n+1}} \prod_{j \in J_v} 2^{\tilde{h}_j - h_j} \int dk\, \alpha\, d\alpha\, d\boldsymbol{\omega}\, \tau \left(\prod_{j \neq j_0} 2^{-2h_j} d\boldsymbol{\omega}_j \right) \\ \times \exp[-(N+\alpha) k^2] \exp\left[+\mathbf{w}^2/(\alpha+N) - \sum_p \boldsymbol{\Omega}_p^2 \right] \\ \times \prod_j \{ \varepsilon_j [ik_0 + \boldsymbol{\omega}_j \mathbf{k} - \boldsymbol{\omega}_j \mathbf{w}/(\alpha+N)] + \boldsymbol{\omega}_j \boldsymbol{\Omega}_j \} \\ \times \prod_{J_\alpha} \{ \varepsilon_j [\boldsymbol{\omega}_j \mathbf{k} - \boldsymbol{\omega}_j \mathbf{w}/(\alpha+N)] + \boldsymbol{\omega}_j \boldsymbol{\Omega}_j \} \prod_{J_\zeta} (i\varepsilon_j k_0) \end{aligned} \tag{11.18}$$

where we denote J_ν, J_α, J_ζ, respectively, the set of lines ending in a vertex with running coupling of type ν, α, ζ.

We rewrite $\boldsymbol{\omega}_j$ as $(\boldsymbol{\omega}_j - \boldsymbol{\omega}) + \boldsymbol{\omega}$ and the ik_0 in the last product as $(ik_0 + \boldsymbol{\omega}\mathbf{k}) - \boldsymbol{\omega}\mathbf{k}$. Then we develop the sums inside the products to isolate the contributions of the forms

$$\begin{aligned} &(ik_0 + \boldsymbol{\omega}\mathbf{k}), \qquad (\boldsymbol{\omega} - \boldsymbol{\omega}_j)\mathbf{k}, \qquad \boldsymbol{\omega}\mathbf{w} \\ &(\boldsymbol{\omega} - \boldsymbol{\omega}_j)\mathbf{w}, \qquad 2^{h_j}\boldsymbol{\Omega}_j^2 = -2\boldsymbol{\omega}_j \boldsymbol{\Omega}_j = 2\boldsymbol{\omega}\boldsymbol{\Omega}_j \end{aligned} \tag{11.19}$$

After further algebra we find that (11.18) is a sum of terms equal, up to a sign, to

$$M^{n+1}2^{h-h_{n+1}}\left(\prod_{j\in J_v}2^{\tilde h_j-h_j}\right)\int \alpha\, d\alpha\, dk\, d\omega\, \tau\left(\prod_{j\neq j_0}2^{-2h_j}\, d\omega_j\right)$$
$$\times\exp[-(N+\alpha)\,k^2]\exp\left[+\mathbf{w}^2/(\alpha+N)-\sum_p \mathbf{\Omega}_p^2\right]$$
$$\times\left[\prod_{j\in X_1}\varepsilon_j(ik_0+\boldsymbol{\omega}\mathbf{k})\right]\left[\prod_{j\in X_2}\varepsilon_j\mathbf{k}(\boldsymbol{\omega}-\boldsymbol{\omega}_j)\right]$$
$$\times\left[\prod_{j\in X_3}\varepsilon_j(\boldsymbol{\omega}-\boldsymbol{\omega}_j)\mathbf{w}/(\alpha+N)\right]$$
$$\times\left[\prod_{j\in X_4}\varepsilon_j\boldsymbol{\omega}\mathbf{w}/(\alpha+N)\right]\prod_{j\in X_5}[2^{h_j}\mathbf{\Omega}_j^2]\left[\prod_{j\in X_6}\varepsilon_j\boldsymbol{\omega}\mathbf{k}\right] \tag{11.20}$$

where X_q are subsets of the set of the $n+1$ indices j, and each index in X_q can have a multiplicity: in the latter case this simply means that the term with index j has to be raised to a power equal to the multiplicity.

The expansion of the products in (11.18) allows us to find the rules to construct the sets X_q. In fact, by inspection of (11.18), we see that we can define five suitable disjoint subsets of the set of the $n+1$ indices denoted J_1^s, $s=1,\ldots,5$, and similarly J_α can be broken into five sets J_2^s, and J_ζ into two sets J_3^s, so that

$$\begin{aligned} X_1&=J_1^1\cup J_3^1, & X_4&=J_1^4\cup J_2^3\\ X_2&=J_1^2\cup J_2^1, & X_5&=J_1^5\cup J_2^4\\ X_3&=J_1^3\cup J_2^2, & X_6&=J_2^5\cup J_3^2 \end{aligned} \tag{11.21}$$

and each point in X_q has multiplicity equal to the number of times it appears in distinct sets in the rhs.

The discussion of (11.19) is simpler than one might fear. The quadratic form in the $\mathbf{\Omega}$'s in the exponential is bounded below by $\alpha/(\alpha+N)$. This means that after performing the k integrals we bound $\mathbf{\Omega}_j$ by $[(\alpha+N)/\alpha]^{1/2}$.

The evaluation of the k_0 integral is done by the well-known formula for the Hermite polynomials $H_p(k_1)$ [with weight $\exp(-k_1^2)$]:

$$:k_1^p:\equiv H_p(k_1)=\int e^{-k_0^2}(ik_0+k_1)^p\,\frac{dk_0}{\pi^{1/2}} \tag{11.22}$$

where the polynomial normalization is that the integral of H_p^2 with weight $\exp(-k_1^2)$ is $2^{-p}p!$ (Wick rule).

Hence, changing the variable $(\alpha+N)^{1/2}k \to k$, setting $\eta = (\alpha+N)/\alpha$, and using $\omega(\omega-\omega') = (\omega-\omega')^2/2$, we can rewrite (11.20) as

$$
\begin{aligned}
& M^{n+1}\left[2^{h-h_{n+1}}\prod_{j\in J_v}(2^{\tilde{h}_j-h_j})\right]\int \frac{\tau\alpha\, d\alpha\, d\mathbf{k}}{(\alpha+N)^2}\, d\omega\, [\exp(-\mathbf{k}^2)] \\
& \times\left[\prod_{j\neq j_0} 2^{-2h_j}\, d\omega_j\right]\exp\left(\frac{\mathbf{w}^2}{\alpha+N}-\sum_p \mathbf{\Omega}_p^2\right) \\
& \times\left[:\prod_{j\in X_1}\frac{\varepsilon_j\omega\mathbf{k}}{(\alpha\eta)^{1/2}}:\right]\left[\prod_{j\in X_2}\frac{\varepsilon_j\mathbf{k}(\omega-\omega_j)}{(\alpha\eta)^{1/2}}\right] \\
& \times\left[\prod_{j\in X_3}\varepsilon_j 2^{h_j}\frac{(\mathbf{\Omega}_j)}{(\alpha\eta)^{1/2}}\frac{\mathbf{w}}{(\alpha\eta)^{1/2}}\right] \\
& \times\left[\prod_{j\in X_4}\frac{\varepsilon_j\sum_p 2^h\mathbf{\Omega}_p^2}{2\alpha\eta}\right]\left[\prod_{j\in X_5}2^{h_j}\eta\frac{\mathbf{\Omega}_j^2}{2\eta}\right]\left[\prod_{j\in X_6}\frac{\varepsilon_j\omega\mathbf{k}}{(\alpha\eta)^{1/2}}\right] \qquad (11.23)
\end{aligned}
$$

It is easy to check that, after performing the integral over $\mathbf{k}$, using a suitable fraction of the quadratic form in the exponent, and denoting by c_0 a suitable constant:

1. If X_3 is not empty, we can bound the product over X_3 by $(\prod_{j\in X_3} c_0 2^{h_j}\alpha^{-1})\,|X_3|!$.
2. If X_4 is not empty, we can bound the product over X_4 by $(2^h c_0\alpha^{-1})^{|X_4|}\,|X_4|!$.
3. If X_5 is not empty, we can bound the product over X_5 by $\prod_{j\in X_5} 2^{h_j}(\alpha+N)/\alpha$.

Moreover, it is important to remark that:

4. At least one of X_1 and X_4 is not empty. See (11.18) and use that the j_0 factor contributes only to the products over X_1 and X_4.
5. In performing the $\mathbf{k}$ integral, the Wick contractions (which we use to evaluate the Gaussian integrals) involving X_1 and X_2 or X_2 and X_6 or X_2 and X_2 itself can be bounded [again using a fraction of the quadratic form in the exponential and the fact that such contractions are proportional to $(\omega-\omega_j)\omega = (\omega-\omega_j)^2/2$ or to $(\omega-\omega_j)^2$] by a factor $2^{2h_j}/\alpha$ per contraction. But if we consider the contractions between X_1 and X_6 we only get $1/\alpha$ per contraction.

Then we see that we always find a factor $2^{h'}/\alpha$ at least for each of the $\leqslant(|X_1|+|X_2|+|X_3|)!^{1/2}$ Wick contributions to the Gaussian integral, provided we discard the only possibility which, by the analysis of the above items, does not provide a factor $2^{h'}/\alpha$, namely when all the contractions involve X_1 and X_6 and $X_2, X_3, X_4, X_5 = \varnothing$: but it is easy to see that in this

case $|X_1| > |X_6|$ and the integral is identically zero by the orthogonality of the Hermite polynomials.

We are left with bounding the ω_j integral: for it we still have the quadratic form to use. By splitting it in six equal parts, five will be used for the bounds in the above items. Then the remaining sixth part will be used to perform the integral over the ω_j and to bound it by $[(\alpha+N)/\alpha]^n \leqslant (N+1)^n$. From the bound of the X_5 factors we also get at most $[(\alpha+N)/\alpha]^n \leqslant (N+1)^n$. Hence the total expression (11.14) is bounded by

$$M^{n+1} C^n \left[2^{h-h_{n+1}} \prod_{j \in J_v} (2^{\bar{h}_j - h_j}) \right] n! 4^{2^{h'}} \tag{11.24}$$

At this point we go back to Section 10 and check that the term in square brackets is

$$\prod_v 2^{-(h_v - h_{v'})(\rho_v - z_v)} \tag{11.25}$$

This is so simply because, having undone all the subtractions, we miss the terms expressing the dimensional gains $z_v \leqslant 2$. Hence we can write (11.24) as (11.13) [recalling that the combinatorial factor $\prod_v 1/s_v!$ was taken out in (11.18)].

It is interesting to remark that, if $d=1$, the above analysis also applies and in some sense works even better. In fact it is easy to see that a one-loop graph in which the quasimomentum is not conserved along the loop is necessarily of order $O(2^{h'})$. Hence, one can consider the case in which the quasimomentum is conserved along the loop; in this case, however, the terms with $\omega - \omega_j$ are missing and we see that the above integral (11.23) simply vanishes identically. Hence, if $d=1$, we do not even have the $n!$ to worry about and we have, in all one-loop graphs, a simple extra factor $2^{h'}$, so that (11.8) holds with $\varepsilon = 1$.

Having treated the graphs with more than one loop and those with just one loop, we are left with the simple case of the chain graphs. Of course such graphs are very trivial compared to the above.

Such graphs must contain one line of scale h with momentum equal to its quasimomentum. It is easy to check, if the two half lines composing the line represent $\psi^{\pm}$ fields, that the propagator of this line generates, by the mechanism discussed in Section 10, an approximate delta function $\delta_h(\omega - \omega')$. Here ω' is the line quasimomentum and ω is the external quasimomentum of the field of minus type. Then $\delta_h(\omega - \omega')$ satisfies

$$\begin{aligned} \delta_h(\omega_1 - \omega) &= \text{const} \cdot 2^{-3h} (\omega - \omega_1) \cdot \omega_1 \int_{1/4}^{1} \frac{d\alpha}{\alpha^2} e^{-\alpha 2^{-2h} (\omega - \omega_1)^2 \beta^2} \\ &\leqslant \text{const} \cdot 2^{-3h} (\omega_1 - \omega)^2 e^{-2^{-2h} (\omega_1 - \omega)^2 \beta^2/4} \\ &= \text{const} \cdot 2^h \bar{\delta}_h(\omega_1 - \omega) \end{aligned} \tag{11.26}$$

where δ_h is a normalized approximate delta function of width proportional to 2^h. We see that in this case we can improve the bound (11.11) by a factor 2^h every time that we have to integrate the above propagator or expressions with the same properties.

A simple analysis shows that this is always the case, with one single exception: namely when the graph is a relevant one and one considers its contribution to the α, ζ running constants. This case can be explicitly computed and shown to give a contribution to the beta function $\pm\beta'' v_h^2 + O(2^h)$.

The above considerations can be summarized by writing explicitly the beta functional in the following form, using the notation λ, α, ζ, ν to mean the string of functions describing the running form factors on scales i shorter than h, $(\lambda_i, \alpha_i, \zeta_i, \nu_i)_{i=h, h+1, \ldots, 0}$:

$$\begin{aligned}
\lambda_{h-1} &= \lambda_h + B_1^{(\geqslant 2)}(\alpha, \zeta, \nu, \lambda) + 2^{\varepsilon h} B_2^{(\geqslant 0; \geqslant 1)}(\alpha, \zeta, \nu; \lambda) \\
\nu_{h-1} &= 2\nu_h + 2^{\varepsilon h} B_3^{(\geqslant 2)}(\alpha, \zeta, \nu, \lambda) \\
\alpha_{h-1} &= \alpha_h + \beta'' \nu_h^2 + 2^{\varepsilon h} B_4^{(\geqslant 2)}(\alpha, \zeta, \nu, \lambda) \\
\zeta_{h-1} &= \zeta_h - \beta'' \nu_h^2 + 2^{\varepsilon h} B_5^{(\geqslant 2)}(\alpha, \zeta, \nu\lambda)
\end{aligned} \tag{11.27}$$

where $B_j^{(\geqslant p; \geqslant q)}(x; y)$ denotes a formal power series in x, y in which the x variables appear to order p or higher, and the y appear to order q or higher; $B_j^{(\geqslant p; [a,b])}(x; y)$ denotes a formal power series in which the x variables appear to order $\geqslant p$ and the y appear only to the orders between a and b; $B_j^{(\geqslant p; q)}(x; y)$ denotes again a formal power series in which the x variables appear to order $\geqslant p$ and the y appear exactly to order q; similarly, $B_j^{(p)}(x)$ denotes a polynomial homogeneous of degree p in x. Here $\varepsilon > 0$ can be taken any number $< 1/4$.

Furthermore, if the effective couplings $\mathbf{v}_h = (\nu_h, \alpha_h, \zeta_h, \lambda_h)$ satisfy

$$|\alpha_h|, |\zeta_h| \leqslant M, \qquad |\nu_h| \leqslant M, \qquad |\lambda_h| \leqslant M, \qquad h = 0, -1, \ldots \tag{11.28}$$

then the mth-order terms of the formal power series B_j can be so arranged that the mth order is described by trees with m endpoints and graphs with m graph elements of the form (8.1), (8.2) in such a way that the contribution to $\mathbf{v}_{h-1}$ due to a tree of type (9.1) and a graph G is bounded by

$$E_{\mathcal{T}} M^m D C^{m-1} \prod_{v \geqslant v_0} 2^{-(h_v - h'_v)\varepsilon\rho_v} \tag{11.29}$$

and the relationship between $\mathbf{v}_{h-1}$ and $\mathbf{v}_h, \ldots, \mathbf{v}_0$ is

$$\begin{aligned}
\mathbf{v}_{h-1} = A\mathbf{v}_h + \sum_{G, \mathcal{T}} \eta_{\{\underline{h}\}}(G, \mathcal{T}) \int 2^{\mu(G)h} \frac{d\boldsymbol{\Omega}}{m!} \\
\times \prod_{v \geqslant v_0} \frac{2^{-(h_v - h_{v'})\varepsilon\rho_v}}{s_v!} \prod_{i \in \text{endlines}} \lambda_{h_i}(\Omega_i)
\end{aligned} \tag{11.30}$$

where Λ is the diagonal matrix $(1, 1, 1, 2)$ and $\eta(G, \vartheta)$ is an interpolation parameter in $[-1, 1]$; the number ρ_v is $\geqslant \varepsilon$ and $\mathbf{\Omega}_i$ are the ω-variables relative to the ith graph element of the graph G; the $\mu(G)$ is 0, ε as described in the above discussion.

The case $d=1$ will be discussed only in the spinless model. In this case (11.8) no longer holds. Besides the remarkable simplification of the absence of the $O(2^{\varepsilon h})$ contributions in the λ part of the beta functional, we find that the beta functional takes the form

$$\begin{aligned}
\lambda_{h-1} &= \lambda_h + B_1^{(\geqslant 2)}(\lambda) + B_2^{(\geqslant 1;\geqslant 1)}(\alpha, \zeta, \nu; \lambda) \\
\nu_{h-1} &= 2\nu_h + B_3^{(\geqslant 1;\geqslant 2)}(\alpha, \zeta, \nu; \lambda) + 2^h B_4^{(\geqslant 2)}(\alpha, \zeta, \nu, \lambda) \\
\alpha_{h-1} &= \alpha_h + B_5^{(\geqslant 2)}(\lambda) + B_6^{(\geqslant 1;\geqslant 2)}(\alpha, \zeta, \nu; \lambda) + 2^h B_7^{(\geqslant 2)}(\alpha, \zeta, \nu, \lambda) \\
\zeta_{h-1} &= \zeta_h + B_5^{(\geqslant 2)}(\lambda) + B_8^{(\geqslant 1;\geqslant 2)}(\alpha, \zeta, \nu; \lambda) + 2^h B_9^{(\geqslant 2)}(\alpha, \zeta, \nu, \lambda)
\end{aligned} \tag{11.31}$$

with the notations of (11.27). The remarkable features are the absence of terms depending only on λ in the relation for ν and the equality of the leading (i.e., not proportional to 2^h) terms dependent only on λ (denoted B_5) in the relations for α and for ζ.

The first remark follows from the observation that only graphs in which there is quasimomentum conservation at *every vertex* can give contributions which cannot be bounded by $O(2^h)$ at least. In fact, the conservation is automatic in the relevant terms with four lines, and nonconservation in two-line subgraphs leads to a very small extra factor in the bounds (because if the momentum is not conserved, then it is necessarily opposite and the Fourier transforms of the propagators decay very fast at large momenta). Considering only graphs with quasimomentum conservation, we see that the oscillating factors due to the exponentials involving the quasimomenta are completely absent from the graph value. Then one can remark that a contribution from a graph with only λ vertices must contain an odd number of inner lines: but the propagators are, up to corrections of $O(2^h)$, odd functions, so that their leading terms do not contribute. The second remark is an easy consequence of the fact that the leading term in the propagator has a symmetry between the space and the time variables.

This concludes our theory of the improved bounds on the beta functional.

It is important, for later applications, to remark that the use of the Wick ordering in the above formalism, although elegant, is by no means essential: and it may become a nuisance when one gets involved with the problem of convergence.

It is possible, and easy, to extend the above theory to the case in

which the effective potential is written as the sum of a part $V_{p,w}$ of degree $\leqslant p$ in the fields, which is written in a Wick ordered form, and a part $V_{p,nw}$ composed of ordinary monomials of degree $> p$.

Taking $p \geqslant 4$, the above formalism remains unchanged with a few minor variations:

(a) The trees have one vertex per each frequency, which may be trivial, i.e., such that no bifurcation occurs: such vertices can be thought of as existing but not explicitly marked in the previous formalism. However, this time the associated truncated expectations on the vertex scale are no longer trivial, because the absence of the Wick ordering permits contractions between lines emerging from the same cluster v, *self-contractions*, at least if the subgraph G_v has more than p external lines.

(b) Due to the possibility of self-contractions, the frequency of a line is no longer determined by the clusters containing its extreme endpoints and one finds it by looking at the smallest cluster containing the line.

(c) All internal lines are *hard* lines, in the sense of Section 9, with the possible exceptions of lines which are internal to clusters with $\leqslant p$ external lines.

The separation order p is taken to be $\geqslant 4$, so that the $\mathscr{L}$, $\mathscr{R}$ operations are still defined as in Section 7. All that is happening is that one has to deal with roughly the same number of graphs bearing a different set of indices and labels: it is a reorganization of a multiple sum. The estimates are clearly identical to the previous ones. The beta function is different, and as remarked in refs. 16 and 17, it may be much better from the point of view of the estimates.

The two beta functionals manifestly agree up to an order p^* and $p^* \to \infty$ if $p \to \infty$.

A deeper remark is that we can even avoid entirely the use of Wick ordering, i.e., take $p = 0$. At first sight one might think that this really changes the beta function. However, a closer look shows that it only affects the beta function with terms of $O(2^h)$ as $h \to -\infty$: the reason is that the propagators $g^{(h)}(x)$ *vanish* when $x = 0$ up to terms of $O(2^h)$.

We shall avoid choosing among the various beta functionals and what we say holds for any choice (only the actual numerical values of some bounds may differ). However, in the analysis of the convergence, which we are able to do only if $d = 1$ and the fermions are spinless, we use the results of ref. 19 and therefore we use the $p = 4$ beta functional.

12. CONSEQUENCES OF THE BOUNDS. THE PERTURBATION THEORY BETA FUNCTIONAL. BOUNDS ON THE SCHWINGER FUNCTIONS NEAR THE FERMI SURFACE

The conclusion of the work of Sections 10 and 11 is thus that, if in $\mathbf{v}_h = (v_h, \alpha_h, \zeta_h, \lambda_h)$, the effective couplings $\mathbf{v}_h$ satisfy

$$|\alpha_h|, |\zeta_h| \leqslant M, \qquad |v_h| \leqslant M, \qquad |\lambda_h| \leqslant M, \qquad h = 0, -1, \ldots \tag{12.1}$$

then $\mathbf{v}_{h-1}$ can be expanded in a formal power series of $\mathbf{v}_h, \mathbf{v}_{h+1}, \ldots$ and the power expansion can be arranged to have the form (11.30).

Since the eigenvalues of A [see comment after (11.10)] are all $\geqslant 1$, the analysis of refs. 16–18 applies and we infer from (11.30) the following results:

1. It is possible to expand $\mathbf{v}_{h+1}$ in powers of $\mathbf{v}_h$:

$$\mathbf{v}_{h+1} = A^{-1}\mathbf{v}_h + \mathbf{B}^h_+(\mathbf{v}_h) + 2^h R_h(\mathbf{v}_h, \mathbf{v}_0) \tag{12.2}$$

where $B^h_+(\mathbf{v})$ (*upward beta function*) is a formal power series:

$$B^h_+(\mathbf{v})(\mathbf{\Omega}) = \sum_{p=1} \sum_{\underline{a}} \int d\mathbf{\Omega}'\, \eta^{\underline{a}}_h(\mathbf{\Omega}, \mathbf{\Omega}')\, C\beta^{|\underline{a}|}\underline{a}!\mathbf{v}^{\underline{a}}(\mathbf{\Omega}') \tag{12.3}$$

where $\underline{a} = (a_\lambda, a_v, a_\alpha, a_\zeta)$ is a multiindex and $\lambda_h(\mathbf{\Omega})$ is regarded as defined and bounded on the unit sphere S^{d-1}, uniformly in the h. The kernel $\eta_{\underline{a}}(\mathbf{\Omega}, \mathbf{\Omega}')$ is bounded also by 1 when $\mathbf{\Omega}$ is regarded as composed of vectors varying on the unit sphere S^{d-1}. The R_h function has the same properties and vanishes with $\mathbf{v}_0$. It is a *new* term compared with the results of refs. 16 and 18: it is present because the initial interaction contains nonvanishing irrelevant terms expressible via the initial pair potential. The shape of the pair potential has to be supposed fixed here once and for all: otherwise, the coefficients of the power series defining R_h also depend on the shape of the potential in a way in which we are not interested here.

2. A similar expansion holds for the general effective potentials on scale h. The effective potential on scale h admits an expansion in $\mathbf{v}_h$ and $\mathbf{v}_0$ with coefficients growing with $m!$ to order m and with kernels of size C_0 (in the sense of Section 9), uniformly bounded by

$$DC^{m-1}m! \tag{12.4}$$

for suitably chosen D, C.

3. The iteration of the expansion (12.2) leads to an expansion of the effective couplings and potentials on all scales $h' \geqslant h$. The coefficients are bounded as in (12.4), but now $\beta^{|\underline{a}|}$ is replaced by $[\beta(h'-h)]^{|\underline{a}|}$. This means that even if it were true that $\mathbf{v}_h \to 0$, we would only have control over the

large-scale behavior of the theory. The short-range behavior is hidden in perturbation theory (as is not surprising).

4. The B terms appearing in the beta function have a detailed structure identical to that in (11.25) with new B functions which depend only on $\mathbf{v}_h$.

In Appendix C we show that the interpretation of the constancy of the effective couplings as $h \to -\infty$ can be that the potential is a delta-like potential in the sense of the Introduction with integral $\propto 2^{(d-1)h}$. It is unlikely that the running constants go to 0 as $h \to -\infty$ faster than logarithmically (i.e., as $1/h$). Hence, working in physical space, we expect to find a potential whose size is at least $2^{-h}/h$, i.e., unboundedly large! This shows the relevance and the interest of the quasiparticle formalism: in this formalism we never meet large functions and we can apply the strategy of using supremum bounds. In the usual representation in Fock space, or in the Euclidean versions of it, one would be forced to use L_1 bounds, which are far harder to master.

In one dimension the Fermi sphere consists of two points. All the functions of the quasimomenta become trivially defined on a space of two points and the bounds can be revisited and trivially improved. It turns out that, not surprisingly, the theory of the beta function becomes the same as the theory of the Gross–Neveu model.[19,23] One rigorously finds that the series in (12.3) converge if M [see (12.1)] is small enough. Hence the beta function exists as a holomorphic function of the running couplings with a positive radius of convergence.

All the results found so far have been derived for spinless fermions, but the introduction of spin would just add a few more indices to our expressions; this is no longer true for the results of Sections 14 and 15.

In the next section we study the beta functional to second order in the general spinless case; in Section 15 we study in detail, for $d=1$, the beta function, introduce a new beta functional to describe anomalous Fermi surfaces, and show that one-dimensional spinless systems have an anomalous Fermi surface.

We conclude this section by analyzing the connection between the definitions of Section 5 and the bounds of Sections 10 and 11 in the case $d=3$. We use here the notations of Sections 10 and 11.

In studying the Schwinger functions, one can still make use of the notion of C_0 size: however, we have to consider more general graphs in which vertices $y_1, y_2, \ldots$ with only one emerging line of type ψ^+ or ψ^- are allowed, with a form factor $e^{\pm iy_j k_j}$ (the k_j are arbitrary momenta); furthermore, such vertices must be on scales $h_1, h_2, \ldots$ between 0 and h and the lines corresponding to them have to be ignored in the localization procedures.

Then one is interested in the graphs generated by such vertices, arbitrary trees, and *no external lines*. It is clear that (by the relation determined in Section 3 between Schwinger functions and the effective potential) the C_0 size of such graphs evaluated on a given tree is a bound on the Fourier transform of the contribution to the Schwinger function from the considered graph and tree.

We shall consider in particular the graphs with m graph elements of the types in (8.1) and (8.2), and two one-line vertices of type ψ^+, ψ^-. This is the set of graphs relevant for the pair Schwinger function, i.e., relevant to calculate the function $w'_h(k)$ introduced in (5.33) and (7.14).

If we now suppose that the frequency labels of the lines coming out of the one-line vertices are h_1, $h_2 \geqslant h$, the estimates in Section 10 provide obvious bounds. However, the bounds that one finds are useless if performed naively. This is in fact a major problem: its solution is deeply connected with the underlying symmetry, pointed out several times, generated by the fact that the quasiparticle fields are a redundant description of the particle fields.

Let $\mathscr{G}$ be the set of contributions to the effective potential of degree two in the fields: the irrelevant ones can be divided into those contributing to terms like $\psi^+ S^j$, $D^+ D^-$, $D^+ S$ [see (7.8)]. Given $G \in \mathscr{G}$, its contribution to the pair Schwinger function at external momentum k can be written as

$$\int \hat{g}(k-\omega', \boldsymbol{\omega}')\, W_G(k, \omega', \omega)\, \hat{g}(k-\omega, \boldsymbol{\omega})\, d\omega\, d\boldsymbol{\omega}' \tag{12.5}$$

where $\hat{g}(k, \boldsymbol{\omega})$ is the Fourier transform of the free propagator $g^{(\geqslant h)}(x, \boldsymbol{\omega})$, which is uniformly bounded from below by $1/(|k_0| + p_F\,|\Delta|)$, if $k = (k_0, \mathbf{k})$ and $\mathbf{k} \equiv \boldsymbol{\omega}_0 p_F(1+\Delta)$.

For instance, a graph contributing to the terms $\psi^+_{z\boldsymbol{\omega}'} S^1_{yx\boldsymbol{\omega}}$ of the effective potential will give a contribution to the pair Schwinger function of the form

$$\int e^{ik(z'-y')} g(z'-z, \boldsymbol{\omega}')\, e^{ip_F\boldsymbol{\omega}'(\mathbf{z}-\mathbf{z}')} V_G(z-y,\, y-x, \boldsymbol{\omega}, \boldsymbol{\omega}')(x-y)^2$$

$$\times\, g''(y-y'+t(x-y), \boldsymbol{\omega})\, e^{ip_F\boldsymbol{\omega}(\mathbf{y}'-\mathbf{y})} t\, dt\, d\boldsymbol{\omega}\, d\boldsymbol{\omega}'\, dx\, dy\, dz\, dy' \tag{12.6}$$

where t is an interpolation parameter, V_G is a suitable kernel, and $(x-y)^2$ is a two-index tensor contracted with the tensor of the second derivatives g''; (12.6) can be rewritten as an integral, over the $\boldsymbol{\omega}$, $\boldsymbol{\omega}'$ variables and over the interpolation parameter t, of the expression

$$\hat{g}(k-\omega', \boldsymbol{\omega}')\, V_G(k, t(k-\boldsymbol{\omega}), \boldsymbol{\omega}, \boldsymbol{\omega}')\, \hat{g}''(k-\omega, \boldsymbol{\omega})$$

$$\equiv \hat{g}(k-\omega', \boldsymbol{\omega}')[\hat{V}_G(k, t(k-\boldsymbol{\omega}), \boldsymbol{\omega}, \boldsymbol{\omega}')_{ij}\,(k-\omega)_i\,(k-\omega)_j]\, \hat{g}(k-\omega, \boldsymbol{\omega}) \tag{12.7}$$

where $\hat{V}_G$ denotes the Fourier transform of $(x-y)^2 V_G$, so that W_G [see (12.5)] is defined by the term in square brackets.

The sum over the *full* set of contributions is

$$\sum_{G \in \mathscr{G}} W_G(k, \boldsymbol{\omega}, \boldsymbol{\omega}') \equiv w_h(k) \tag{12.8}$$

and is $\boldsymbol{\omega}$, $\boldsymbol{\omega}'$ *independent* by the symmetry property pointed out in Sections 5 and 7.

Therefore, we can choose, in evaluating w'_h, which is our main concern here, $\boldsymbol{\omega}$, $\boldsymbol{\omega}'$ arbitrarily. We take them as $\boldsymbol{\omega} = \boldsymbol{\omega}' = \boldsymbol{\omega}_0$ if $k = (k_0, \mathbf{k})$ and $\mathbf{k} = p_F(1+\Delta)\,\boldsymbol{\omega}_0$, so that $|\Delta|\, p_F = ||\mathbf{k}| - p_F|$. We deduce that a bound on the contribution $\hat{g}(k)^2 w'_h(k)$ to the Fourier transform of the Schwinger function from the graphs G associated with the terms $\psi^+ S^1$ in the effective potential can be computed, bounding $(k-\omega_0)^2$ by $(p_F^2 |\Delta|^2 + k_0^2)$ and $|\hat{g}(k-\omega, \boldsymbol{\omega})|^2$ by $1/(p_F^2 |\Delta|^2 + k_0^2)$, by

$$\begin{aligned}
&\int dt\, d\boldsymbol{\omega}'\, d\boldsymbol{\omega}\, |\hat{g}(k+\omega', \boldsymbol{\omega}')|\, |\hat{V}_G(k, t(k-\boldsymbol{\omega}_0), \boldsymbol{\omega}_0, \boldsymbol{\omega}_0)| \\
&\qquad \times |\hat{g}(k+\omega, \boldsymbol{\omega})|\, (p_F^2 \Delta^2 + k_0^2) \\
&\leqslant 2^{-h} \sum_{h_v \geqslant h} \prod_{v \geqslant v_0} 2^{-(h_v - h_{v'})\rho_v} \\
&\leqslant \text{const} \cdot 2^{-h}
\end{aligned} \tag{12.9}$$

This simply follows by remarking that $2^h |\hat{V}_G(\cdots)|$ is bounded by the C_0 size, in the sense of Sections 9 and 10, of the graph with two external lines obtained from G by regarding the two lines ending in the two one-line vertices as external lines.

Of course, once we fix the *gauge* $\boldsymbol{\omega} = \boldsymbol{\omega}' = \boldsymbol{\omega}_0$ we are not allowed to change it when estimating the contributions of the other graphs: but the same choice simply works for all the others, too. Hence the part of order m in the running form factors of $w'_h(k)/(k_0^2 + p_F^2 \Delta^2)$ can be thought of as generated by the appropriate graphs, receiving a contribution bounded by $C^m 2^{-h}$ from each of them.

The above conclusion can be improved using the results of Section 11, where it is shown that there is, in the estimate of the size of a graph with two external lines, a gain of a factor $2^{h\varepsilon}$, where ε is a small enough ($\leqslant 1/4$) positive pre-fixed number [see (11.8)]. Hence we get the bound

$$2^{-(1-\varepsilon)h} E_{\mathscr{G}} \sum_{h_v \geqslant h} \prod_{v \geqslant v_0} 2^{-(h_v - h_{v'})\varepsilon\rho_v} < \text{const} \cdot 2^{-(1-\varepsilon)h} \tag{12.10}$$

The result of this analysis is that $w'_h(k)/(p_F^2 \Delta^2 + k_0^2)$ can be bounded, to order m in the running form factors, by

$$m!\, C^m \cdot \text{const} \cdot 2^{-(1-\varepsilon)h} \tag{12.11}$$

for any (pre-fixed) $\varepsilon < 1/4$.

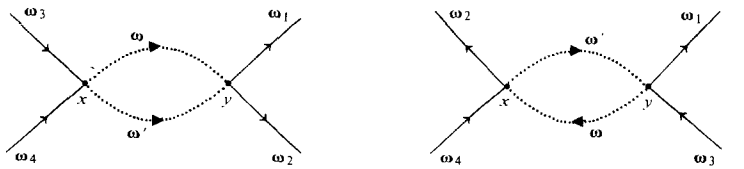
In other words, the contribution of the irrelevant terms in the effective potential to the Fourier transform of the two-point Schwinger function with infrared cutoff at length scale $(2^h p_0)^{-1}$ is much smaller than $(|k_0| + ||\mathbf{k}| - p_F|)^{-1}$ for all momenta in the range $2^{(1-\varepsilon)h} p_0 > |k_0| + ||\mathbf{k}| - p_F| > 2^h p_0$. The latter are the momentum sizes corresponding to the scales between $2^{-(1-\varepsilon)h} p_0^{-1}$ and $2^{-h} p_0^{-1}$. Hence, such contributions provide, in this range, singular corrections small compared to the linear singularity, i.e., of type $1/(|k_0| + ||\mathbf{k}| - p_F|) = (2^h p_0)^{-1}$, of the free propagator.

13. THE BETA FUNCTION TO SECOND ORDER. BASIC CALCULATIONS FOR $d = 1, 3$

In this section we describe the computations necessary to write the beta function truncated to second order in the running form factors λ_h, using the localization operator $\mathscr{L}$ introduced in Section 7 and modified in Section 11. Note that we have shown in Section 11 that, if $d = 3$, these are the only contributions of $O(1)$ to the beta function involving only the form factor λ, if the conditions in (12.1) are assumed. This not the case if $d = 1$: the latter case will be reexamined in Section 15. It is also easy to check that the computed terms are identical if one uses the definition (7.5) of $\mathscr{L}$ or the one adopted later [see Section 11, remark following (11.7)].

We shall also compute the second-order contribution to the beta function using the alternative localization operator $\mathscr{L}_C$, also introduced in Section 11; we shall use it in the heuristic discussion of Section 14.

We compute, therefore, the contributions to the λ equation of the two graphs



(13.1)

which we call the *direct* graph and the *exchange* graph.

We choose x as the *localization* point and suppose that the two vertices represent

$$-\int dx \prod_{i=1}^{4} d\boldsymbol{\omega}_i \, \tilde{\lambda}(\boldsymbol{\omega}_1, \boldsymbol{\omega}_2, \boldsymbol{\omega}_3, \boldsymbol{\omega}_4) \, e^{ip_F(\omega_1+\omega_2-\omega_3-\omega_4)x} \times :\psi^{+}_{x\boldsymbol{\omega}_1}\psi^{+}_{x\boldsymbol{\omega}_2}\psi^{-}_{x\boldsymbol{\omega}_3}\psi^{-}_{x\boldsymbol{\omega}_4}: \tag{13.2}$$

and, in the case in which we study the beta function corresponding to the localization $\mathscr{L}_C$ introduced in Section 11, we shall suppose that λ has the form of a function in the range of the operator P_C [see (11.5)]:

$$\tilde{\lambda}(\boldsymbol{\omega}_1, \boldsymbol{\omega}_2, \boldsymbol{\omega}_3, \boldsymbol{\omega}_4) = \tfrac{1}{4}[\lambda(\boldsymbol{\omega}_1; \boldsymbol{\omega}_3) - \lambda(\boldsymbol{\omega}_1; \boldsymbol{\omega}_4) - \lambda(\boldsymbol{\omega}_2; \boldsymbol{\omega}_3) + \lambda(\boldsymbol{\omega}_2; \boldsymbol{\omega}_4)]$$
$$\lambda(\boldsymbol{\omega}_1; \boldsymbol{\omega}_3) \equiv \tilde{\lambda}(\boldsymbol{\omega}_1, -\boldsymbol{\omega}_1, \boldsymbol{\omega}_3, -\boldsymbol{\omega}_3) \tag{13.3}$$

The latter form for $\tilde{\lambda}(\boldsymbol{\omega}_1,...)$ is not general: we see easily, however, that if $d=1$, the antisymmetry properties of $\tilde{\lambda}$ in $(1,2) \leftrightarrow (2,1)$ and $(3,4) \leftrightarrow (4,3)$ and the symmetry in $(1,2) \leftrightarrow (3,4)$ imply that $\tilde{\lambda}$ must have the form (13.3) with $\lambda(\boldsymbol{\omega}_1, \boldsymbol{\omega}_3)$ odd in $\boldsymbol{\omega}_1$ as well as in $\boldsymbol{\omega}_3$.

In the general case a function with the symmetry properties of $\tilde{\lambda}$ can be written in the form (13.3) plus a remainder consisting of terms vanishing at the points $\boldsymbol{\omega}_2 = -\boldsymbol{\omega}_1$ or $\boldsymbol{\omega}_3 = -\boldsymbol{\omega}_4$. Following the notation of Section 11, we call the two terms, respectively, $P_C\tilde{\lambda}$ and $(1-P_C)\tilde{\lambda}$ [see (11.5)]. We call the function $\lambda(\boldsymbol{\omega}_1; \boldsymbol{\omega}_3)$ the coupling between the two *Cooper pairs* $\pm\boldsymbol{\omega}_1$ and $\pm\boldsymbol{\omega}_3$.

The function $\lambda(\boldsymbol{\omega}_1; \boldsymbol{\omega}_3)$ is a complete description of the local part of the quasiparticle interaction if $d=1$ because, as remarked above, $(1-P_C)\tilde{\lambda}=0$ in this case. In the heuristic discussion of Section 14, we shall give arguments based on the idea that if $d>1$, the local part of the interaction is not entirely relevant and the part $(1-P_C)\tilde{\lambda}$ can in fact be put in the irrelevant effective potential, thereby reducing the problem to a simpler one described by a *reduced* beta function.

The evaluations of the beta function to second order using $\mathscr{L}$ or $\mathscr{L}_C$ are trivially related. Assume that the second-order beta function for the $\mathscr{L}$ localization has been computed expressing the variation $\lambda' - \lambda$ of $\lambda(\boldsymbol{\omega}_1, \boldsymbol{\omega}_2, \boldsymbol{\omega}_3, \boldsymbol{\omega}_4)$ as a sum of two terms $\delta_i(\boldsymbol{\omega}_1, \boldsymbol{\omega}_2, \boldsymbol{\omega}_3, \boldsymbol{\omega}_4)$. Then we can get the beta function for the localization $\mathscr{L}_C$ simply by using a four-argument $\tilde{\lambda}$ function of the form (13.3) and evaluating the result at $(\boldsymbol{\omega}_1, \boldsymbol{\omega}_2, \boldsymbol{\omega}_3, \boldsymbol{\omega}_4)$ in the Cooper pair configuration $(\boldsymbol{\omega}_1, -\boldsymbol{\omega}_1, \boldsymbol{\omega}_3, -\boldsymbol{\omega}_3)$; hence, the variation of the running form factors associated with $\mathscr{L}_C$ will be

$$\lambda'(\boldsymbol{\omega}; \boldsymbol{\omega}') = \lambda(\boldsymbol{\omega}; \boldsymbol{\omega}') + \delta_1(\boldsymbol{\omega}; \boldsymbol{\omega}') + \delta_2(\boldsymbol{\omega}; \boldsymbol{\omega}') \tag{13.4}$$

where $\delta_i(\boldsymbol{\omega}; \boldsymbol{\omega}')$ denotes $\delta_i(\boldsymbol{\omega}, -\boldsymbol{\omega}, \boldsymbol{\omega}', -\boldsymbol{\omega}')$.

If we shorten $:\psi^{+}_{x\omega_1}\psi^{+}_{x\omega_2}\psi^{-}_{x\omega_3}\psi^{-}_{x\omega_4}:$ to $:\psi_{1234}:$, the values of δ_1 and δ_2 are extracted from the values of the above two graphs, which we write in the direct graph case as

$$\begin{aligned}
&\frac{4}{2!}\int dx \prod_{i=1}^{4} d\omega_i\, e^{ip_F(\omega_1+\omega_2-\omega_3-\omega_4)x} :\psi_{1234}: \\
&\quad\times \int d\omega\, d\omega'\, dz\, \tilde\lambda(\omega_1,\omega_2,\omega,\omega')\,\tilde\lambda(\omega',\omega,\omega_3,\omega_4)\, e^{-ip_F(\omega+\omega'-\omega_1-\omega_2)z} \\
&\quad\times 2^{2h}[\, g^{(h)}(2^h p_0 z,\omega)\, g^{(h)}(2^h p_0 z,\omega') \\
&\quad + 2g^{(h)}(2^h p_0 z,\omega)\, g^{(<h)}(2^h p_0 z,\omega')] \qquad (13.5)
\end{aligned}$$

and in the exchange graph case

$$\begin{aligned}
&-\frac{4^2}{2!}\int dx \prod_{i=1}^{4} d\omega_i\, e^{-ip_F(\omega_1+\omega_2-\omega_3-\omega_4)x} \\
&\quad\times :\psi_{1234}: \Big\{ \int d\omega\, d\omega'\, dz\, \tilde\lambda(\omega_1,\omega,\omega_3,\omega') \\
&\quad\times \tilde\lambda(\omega_2,\omega',\omega_4,\omega)\, 2^{2h} G_h(2^h p_0 z,\omega,\omega')\, e^{-ip_F(\omega'-\omega-\omega_1+\omega_3)z} \Big\}_{AS} \qquad (13.6)
\end{aligned}$$

where the AS label means that the expression inside the brackets has to be antisymmetrized in ω_1, ω_2 and ω_3, ω_4 separately. No such antisymmetrization is necessary on the first term. The G_h function in (13.6) turns out to be identical to the term in square brackets in (13.5), to leading order as $h \to -\infty$, using the fact that $g^{(h)}(z,\omega)$ is odd in z at the leading order and in fact can be computed from the formulas

$$\begin{aligned}
g^{(h)}(\xi,\omega) &= c_0(\tau - \beta^{-1} i\omega\xi)\,\gamma_0(\xi^2) \\
g^{(<h)}(\xi,\omega) &= c_0(\tau - \beta^{-1} i\omega\xi) \sum_{n=-\infty}^{-1} 2^{2n}\gamma_0(2^{2n}\xi^2)
\end{aligned} \qquad (13.7)$$

where c_0 is a constant [see (A9)], $\xi = (\tau, \xi)$, and $\xi^2 = \tau^2 + \xi^2/\beta^2$. This is not essential in what follows, but it greatly simplifies the calculations. The main problem is that, if the subleading terms are included, the G_h function appearing in (13.6) is not equal to the square bracket terms in (13.5), and this leads to minor corrections in the analysis below.

We see from (13.7) that, if (A9) and (A10) are taken into account and if we set

$$\begin{aligned}
\sigma(\alpha) = \frac{c_0^2}{\alpha^2}\int_{1/4}^{1} \frac{d\alpha_1\, d\alpha_2}{\alpha_1^2\alpha_2^2}\Big[&\delta\left(\frac{1}{\alpha}-\frac{1}{\alpha_1}-\frac{1}{\alpha_2}\right) \\
&+ 2\sum_{n=-\infty}^{-1} 2^{2n}\delta\left(\frac{1}{\alpha}-\frac{1}{\alpha_1}-\frac{2^{2n}}{\alpha_2}\right)\Big]
\end{aligned} \qquad (13.8)$$

then

$$G_h(\xi, \boldsymbol{\omega}, \boldsymbol{\omega}') = (\tau - i\beta^{-1}\boldsymbol{\omega}\xi)(\tau - i\beta^{-1}\boldsymbol{\omega}'\xi)\,\gamma(\xi^2)$$
$$\gamma(z) = \int_{1/2}^{2} d\alpha\, \sigma(\alpha)\, e^{-z/4\alpha} \tag{13.9}$$

and the Fourier transform of G_h at zero time momentum appearing in (13.5) and (13.6) has the form, evaluated at the space momentum $\boldsymbol{\Omega}$ and for $d = 3$,

$$\hat{G}_h(\boldsymbol{\Omega}, \boldsymbol{\omega}, \boldsymbol{\omega}') \equiv \int d\xi\, e^{i\boldsymbol{\Omega}\xi} G_h(\xi, \boldsymbol{\omega}, \boldsymbol{\omega}')$$
$$= 4(4\pi)^2 \beta^3 \int \alpha^3 \sigma(\alpha)\, d\alpha$$
$$\times \left(\frac{(\boldsymbol{\omega} - \boldsymbol{\omega}')^2}{4} + \alpha\beta^2 \boldsymbol{\omega} \cdot \boldsymbol{\Omega} \boldsymbol{\omega}' \cdot \boldsymbol{\Omega} \right) e^{-\alpha\beta^2\boldsymbol{\Omega}^2} \tag{13.10}$$

so that we can rewrite the coefficient of the Wick monomial in the ψ in the direct graph (13.5) more explicitly as

$$2p_0^{-(d+1)} \int 2^{-(d-1)h} \hat{G}_h(2^{-h}(p_F/p_0)\boldsymbol{\Omega}, \boldsymbol{\omega}, \boldsymbol{\omega}')$$
$$\times \tilde{\lambda}(\boldsymbol{\omega}_1, \boldsymbol{\omega}_2, \boldsymbol{\omega}, \boldsymbol{\omega}')\, \tilde{\lambda}(\boldsymbol{\omega}', \boldsymbol{\omega}, \boldsymbol{\omega}_3, \boldsymbol{\omega}_4)\, d\boldsymbol{\omega}\, d\boldsymbol{\omega}' \tag{13.11}$$

with $\boldsymbol{\Omega} = \boldsymbol{\omega}_1 + \boldsymbol{\omega}_2 - \boldsymbol{\omega} - \boldsymbol{\omega}'$; and in the exchange graph case

$$-\frac{4}{2} \int 2^{-(d-1)h}\, d\boldsymbol{\omega}\, d\boldsymbol{\omega}'\, p_0^{-(d+1)}$$
$$\times [\hat{G}_h(2^{-h}(p_F/p_0)(\boldsymbol{\omega} - \boldsymbol{\omega}' + \boldsymbol{\omega}_1 - \boldsymbol{\omega}_3), \boldsymbol{\omega}, \boldsymbol{\omega}')$$
$$\times \tilde{\lambda}(\boldsymbol{\omega}_1, \boldsymbol{\omega}, \boldsymbol{\omega}_3, \boldsymbol{\omega}')\, \tilde{\lambda}(\boldsymbol{\omega}_2, \boldsymbol{\omega}', \boldsymbol{\omega}_4, \boldsymbol{\omega})$$
$$- \hat{G}_h(2^{-h}(p_F/p_0)(\boldsymbol{\omega} - \boldsymbol{\omega}' + \boldsymbol{\omega}_2 - \boldsymbol{\omega}_3), \boldsymbol{\omega}, \boldsymbol{\omega}')$$
$$\times \tilde{\lambda}(\boldsymbol{\omega}_2, \boldsymbol{\omega}, \boldsymbol{\omega}_3, \boldsymbol{\omega}')\, \tilde{\lambda}(\boldsymbol{\omega}_1, \boldsymbol{\omega}', \boldsymbol{\omega}_4, \boldsymbol{\omega})$$
$$- \hat{G}_h(2^{-h}(p_F/p_0)(\boldsymbol{\omega} - \boldsymbol{\omega}' + \boldsymbol{\omega}_1 - \boldsymbol{\omega}_4), \boldsymbol{\omega}, \boldsymbol{\omega}')$$
$$\times \tilde{\lambda}(\boldsymbol{\omega}_1, \boldsymbol{\omega}, \boldsymbol{\omega}_4, \boldsymbol{\omega}')\, \tilde{\lambda}(\boldsymbol{\omega}_2, \boldsymbol{\omega}', \boldsymbol{\omega}_3, \boldsymbol{\omega})$$
$$+ \hat{G}_h(2^{-h}(p_F/p_0)(\boldsymbol{\omega} - \boldsymbol{\omega}' + \boldsymbol{\omega}_2 - \boldsymbol{\omega}_4), \boldsymbol{\omega}, \boldsymbol{\omega}')$$
$$\times \tilde{\lambda}(\boldsymbol{\omega}_2, \boldsymbol{\omega}, \boldsymbol{\omega}_4, \boldsymbol{\omega}')\, \tilde{\lambda}(\boldsymbol{\omega}_1, \boldsymbol{\omega}', \boldsymbol{\omega}_3, \boldsymbol{\omega})] \tag{13.12}$$

The contribution to the beta functional of the above graphs is

$$
\begin{aligned}
&\tilde\lambda_{h-1}(\omega_1\omega_2\omega_3\omega_4)\\
&\quad=\tilde\lambda_h(\omega_1\omega_2\omega_3\omega_4)\\
&\qquad+2\int 2^{-(d-1)h}p_0^{-(d+1)}\,d\omega\,d\omega'\\
&\qquad\times\hat G_h(2^{-h}(p_F/p_0)(\omega_1+\omega_2-\omega-\omega'),\omega,\omega')\\
&\qquad\times\tilde\lambda_h(\omega_1,\omega_2,\omega,\omega')\,\tilde\lambda_h(\omega',\omega,\omega_3,\omega_4)\\
&\qquad-2\int 2^{-(d-1)h}p_0^{-(d+1)}\,d\omega\,d\omega'\\
&\qquad\times[\hat G_h(2^{-h}(p_F/p_0)(\omega-\omega'+\omega_1-\omega_3),\omega,\omega')\\
&\qquad\times\tilde\lambda_h(\omega_1,\omega,\omega_3,\omega')\,\tilde\lambda_h(\omega_2,\omega',\omega_4,\omega)\\
&\qquad-\hat G_h(2^{-h}(p_F/p_0)(\omega-\omega'+\omega_2-\omega_3),\omega,\omega')\\
&\qquad\times\tilde\lambda_h(\omega_2,\omega,\omega_3,\omega')\,\tilde\lambda_h(\omega_1,\omega',\omega_4,\omega)\\
&\qquad-\hat G_h(2^{-h}(p_F/p_0)(\omega-\omega'+\omega_1-\omega_4),\omega,\omega')\\
&\qquad\times\tilde\lambda_h(\omega_1,\omega,\omega_4,\omega')\,\tilde\lambda_h(\omega_2,\omega',\omega_3,\omega)\\
&\qquad+\hat G_h(2^{-h}(p_F/p_0)(\omega-\omega'+\omega_2-\omega_4),\omega,\omega')\\
&\qquad\times\tilde\lambda_h(\omega_2,\omega,\omega_4,\omega')\,\tilde\lambda_h(\omega_1,\omega',\omega_3,\omega)]
\end{aligned}
\tag{13.13}
$$

To evaluate the contribution (13.4) to the variations δ_1, δ_2 of the running form factors $\lambda(\omega;\omega')$ associated with the $\mathscr{L}_C$ localization, as remarked above, we simply must evaluate (13.11) and (13.12) at the *Cooper pairs*, i.e., for $\omega_2=-\omega_1$, $\omega_4=-\omega_3$. The results are

$$
\begin{aligned}
\delta_1(\omega_1,\omega_3)=&-\int d\omega\,d\omega'\,p_0^{-(d+1)}2^{-(d-1)h}\\
&\times\hat G_h(2^{-h}(p_F/p_0)(\omega+\omega'),\omega,\omega')\\
&\times[\lambda(\omega_1,\omega)\,\lambda(\omega,\omega_3)-\lambda(\omega_1,\omega)\,\lambda(\omega_3,\omega')]
\end{aligned}
$$

$$
\begin{aligned}
\delta_2(\omega_1,\omega_3)=&-4\int d\omega\,d\omega'\,p_0^{-(d+1)}2^{-(d-1)h}\\
&\times[\hat G_h(2^{-h}(p_F/p_0)(\omega-\omega'+\omega_1-\omega_3),\omega,\omega')\\
&\times\tilde\lambda(\omega_1,\omega,\omega_3,\omega')\,\tilde\lambda(-\omega_1,\omega',-\omega_3,\omega)\\
&-\hat G_h(2^{-h}(p_F/p_0)(\omega-\omega'-\omega_1-\omega_3),\omega,\omega')\\
&\times\tilde\lambda(-\omega_1,\omega,\omega_3,\omega')\,\tilde\lambda(\omega_1,\omega',-\omega_3,\omega)]
\end{aligned}
\tag{13.14}
$$

where the symmetry properties of (13.3) and (13.10) have been used to simplify the expressions.

At this point we must distinguish the $d=3$ case from the $d=1$ case. In the rest of this section we continue with $d=3$. The $d=1$ case can be treated in a much more satisfactory way and is deferred to Section 15.

Not surprisingly, it is useful to represent, if $d=3$, the functions in (13.14) and (13.10) via the spherical harmonics in the polar coordinates ϑ, φ, of $\boldsymbol{\omega}$, which we denote with the usual notation, namely $Y_{l,m}(\boldsymbol{\omega}) = P_{l,|m|}(\cos\vartheta)\, e^{im\varphi} C_{l,|m|}$, where

$$\begin{aligned} P_{l,|m|}(x) &= (-1)^{|m|}\,(1-x^2)^{|m|/2}\,\frac{d^{|m|}}{dx^{|m|}}\,P_l(x) \\ P_{l,|m|}(x) &= (-1)^{l+|m|}\,P_{l,|m|}(-x) \\ P_l(x) &= \frac{(-1)^l}{2^l l!}\,\frac{d^l}{dx^l}\,(1-x^2)^l \\ C^2_{l,|m|} &= (2l+1)\,\frac{(n-|m|)!}{(n+|m|)!} \\ \delta(\boldsymbol{\omega}-\boldsymbol{\omega}') &= \sum_{l=0}^{\infty}\sum_{m=-l}^{l} Y_{l,m}(\boldsymbol{\omega})\,\overline{Y_{l,m}(\boldsymbol{\omega}')} \\ (2l+1)\,P_l(\boldsymbol{\omega}\cdot\boldsymbol{\omega}') &= \sum_{m=-l}^{l} Y_{l,m}(\boldsymbol{\omega})\,\overline{Y_{l,m}(\boldsymbol{\omega}')} \end{aligned} \tag{13.15}$$

where (ϑ, φ) are the latitude and azimuth coordinates of $\boldsymbol{\omega}$ in an arbitrarily prefixed frame of reference on the Fermi sphere.

Thus, we define $g(l, h)$ by the expansion

$$\begin{aligned} &p_0^{-(d+1)} 2^{-(d-1)h} \hat{G}_h(2^{-h}(p_F/p_0)(\boldsymbol{\omega}+\boldsymbol{\omega}'), \boldsymbol{\omega}, \boldsymbol{\omega}') \\ &\quad = \sum_{l,m} g(l,h)\, Y_{l,m}(\boldsymbol{\omega})\,\overline{Y_{l,m}(\boldsymbol{\omega}')} \end{aligned} \tag{13.16}$$

where we used the fact that the lhs is a function of $\boldsymbol{\omega}\cdot\boldsymbol{\omega}'$. Moreover, $g(l,h) \to_{h\to-\infty} b(-1)^l$ with $b>0$, because $2^{-(d-1)h}\hat{G}_h$ is an approximate delta function with width 2^h in the variable $\boldsymbol{\omega}+\boldsymbol{\omega}'$. The positivity of b comes out of an explicit calculation; see below.

We can also write

$$\begin{aligned} \lambda(\boldsymbol{\omega}_1, \boldsymbol{\omega}_3) &= \sum_{l=\text{odd},m} \lambda_l\, Y_{l,m}(\boldsymbol{\omega}_1)\,\overline{Y_{l,m}(\boldsymbol{\omega}_3)} \\ &\equiv \sum_{l\,\text{odd}} (2l+1)\,\lambda_l P_l(\boldsymbol{\omega}_1\cdot\boldsymbol{\omega}_3) \\ \delta_i(\boldsymbol{\omega}_1, \boldsymbol{\omega}_3) &= \sum_{l=\text{odd},m} \delta_{i,l}\, Y_{l,m}(\boldsymbol{\omega}_1)\,\overline{Y_{l,m}(\boldsymbol{\omega}_3)} \\ &\equiv \sum_{l\,\text{odd}} (2l+1)\,\delta_{i,l} P_l(\boldsymbol{\omega}_1\cdot\boldsymbol{\omega}_3) \end{aligned} \tag{13.17}$$

where only the odd l's enter into λ, δ, since they are separately odd in ω_1, ω_3 and, furthermore, they are symmetric and rotation invariant in ω_1, ω_3 (the latter property also is needed to explain the l dependence of the coefficients).

Then, for all odd l, the $g(l, h)$ are eventually negative for all negative h large enough in modulus and simultaneously for all l

$$\delta_{1l} = -[g(0, h) - g(l, h)]\, \lambda_l^2 \leqslant 0 \tag{13.18}$$

In fact, $g(0, h) \geqslant |g(l, h)|$, since $|P_l| \leqslant 1$ and

$$\begin{aligned} g(l, h) = 4(4\pi)^2 \beta^3 \int \alpha^3 \sigma(\alpha)\, d\alpha \int 2^{-(d-1)h} [(\omega - \omega')^2/4 \\ + 2^{-2h} (p_F/p_0)^2\, \alpha\beta^2 (1 + \omega \cdot \omega')^2] \\ \times \{\exp[-(\omega + \omega')^2\, \alpha\beta^2 2^{-2h} (p_F/p_0)^2]\}\, P_l(\omega \cdot \omega')\, p_0^{-(d+1)}\, d\omega' \end{aligned} \tag{13.19}$$

Furthermore, if l is odd, one realizes, by changing ω' into $-\omega'$ that $g(l, h)$ is $\leqslant 0$ up to a correction of $O(2^h)$ coming from the second term in the integral in (13.19). The limit as $h \to -\infty$ of the coefficient in the rhs of (13.18) is $-2b$, where $b > 0$ is the limit of $g(0, h)$. One can easily estimate an l-independent value of h such that for all smaller values $|g(l, h)| > b/2$.

The theory of δ_2 is more elaborate. We begin by remarking that [see (13.14)]

$$\delta_2(\omega_1, \omega_3) = \bar{\delta}_2(\omega_1 - \omega_3) - \bar{\delta}_2(\omega_1 + \omega_3) \tag{13.20}$$

where, if $\Omega = \omega - \omega' + \omega_1 - \omega_3$ and $N = \sqrt{\alpha}\, \beta 2^{-h} (p_F/p_0)$, and if the function $\hat{\sigma}$ is

$$\hat{\sigma}(\alpha) = 4^2 (4\pi)^2\, \beta^3 \alpha^3 \sigma(\alpha) / [(\sqrt{\alpha}\, \beta (p_F/p_0)^{d-1}\, p_0^{d+1}]$$

we have

$$\begin{aligned} \bar{\delta}_2(\omega_1 - \omega_3) = -\int \hat{\sigma}(\alpha)\, d\alpha \int d\omega\, d\omega' \\ \times \left\{ N^{d-1} \left[\frac{(\omega - \omega')^2}{4} + N^2 \omega \cdot \Omega \omega' \cdot \Omega \right] e^{-N^2 \Omega^2} \right\} \\ \times \bar{\lambda}(\omega_1, \omega, \omega_3, \omega')\, \bar{\lambda}(-\omega_1, \omega', -\omega_3, \omega) \end{aligned} \tag{13.21}$$

and we realize that $\delta_{2,l} \equiv 2\bar{\delta}_{2,l}$ for all odd l.

Therefore we study the distribution

$$\int \left\{ N^2 \left[\frac{(\boldsymbol{\omega}-\boldsymbol{\omega}')^2}{4} + N^2 \boldsymbol{\omega}\cdot\boldsymbol{\Omega}\boldsymbol{\omega}'\cdot\boldsymbol{\Omega} \right] e^{-N^2\Omega^2} \right\} F(\boldsymbol{\omega}_1\boldsymbol{\omega}_3\boldsymbol{\omega}\boldsymbol{\omega}')\, d\boldsymbol{\omega}\, d\boldsymbol{\omega}' \quad (13.22)$$

where F is a test function, $N \to \infty$, and $d = 3$.

In Appendix B we analyze (13.22), choosing

$$F(\boldsymbol{\omega}_1\boldsymbol{\omega}_3\boldsymbol{\omega}\boldsymbol{\omega}') = \frac{1}{4^2}\left[\lambda(\boldsymbol{\omega}_1,\boldsymbol{\omega}_3) - \lambda(\boldsymbol{\omega}_1,\boldsymbol{\omega}') - \lambda(\boldsymbol{\omega},\boldsymbol{\omega}_3) + \lambda(\boldsymbol{\omega},\boldsymbol{\omega}')\right]$$
$$\times \left[\lambda(\boldsymbol{\omega}_1,\boldsymbol{\omega}_3) + \lambda(\boldsymbol{\omega}_1,\boldsymbol{\omega}) + \lambda(\boldsymbol{\omega}',\boldsymbol{\omega}_3) + \lambda(\boldsymbol{\omega}',\boldsymbol{\omega})\right] \quad (13.23)$$

We show that the leading term as $N \to \infty$ is given by

$$\langle F\rangle\, I(N\Delta) \equiv \langle F\rangle \frac{1}{8}\int_{|y|<\Delta N} \frac{dy}{\Delta N}\left[\frac{1}{2} + \left(y^2 - \frac{1}{2}\right)\boldsymbol{\omega}_1\cdot\boldsymbol{\omega}_3\right] e^{-y^2} \quad (13.24)$$

where, if R_φ denotes the rotation by φ around the axis parallel to $\boldsymbol{\Delta}$,

$$\langle F\rangle \equiv \int \frac{d\varphi}{2\pi} F(\boldsymbol{\omega}_1,\boldsymbol{\omega}_3,R_\varphi\boldsymbol{\omega}_3,R_\varphi\boldsymbol{\omega}_1)$$
$$= \int \frac{d\varphi}{2\pi}\frac{1}{4^2}\left[2\lambda(\boldsymbol{\omega}_1,\boldsymbol{\omega}_3) - \lambda(\boldsymbol{\omega}_3,R_\varphi\boldsymbol{\omega}_3) - \lambda(\boldsymbol{\omega}_1,R_\varphi\boldsymbol{\omega}_1)\right]$$
$$\times \left[2\lambda(\boldsymbol{\omega}_1,\boldsymbol{\omega}_3) + \lambda(\boldsymbol{\omega}_1,R_\varphi\boldsymbol{\omega}_3) + \lambda(\boldsymbol{\omega}_3,R_\varphi\boldsymbol{\omega}_1)\right] \quad (13.25)$$

It also follows that δ_1 and δ_2 satisfy the bounds

$$|\delta_1| \leqslant C(\max \lambda^2) \quad (13.26)$$

$$|\delta_2| \leqslant C(\max \lambda^2)\left[J_0(N\Delta) + \frac{1}{N}\right] \quad (13.27)$$

where

$$J_0(x) = \frac{1}{x}\int_0^x dy\; y^2 e^{-y^2} \leqslant C\frac{x^2}{1+x^3} \quad (13.28)$$

The above bounds are not sufficient for the theory of the flow generated by the full beta function, nor that of the *reduced* one in (13.4). One can find some better bounds by making more use of the form of δ_2; however, we cannot really solve the problem even in the case that all the higher orders are neglected. We give a heuristic discussion on this point in Section 14.

In Section 15 we analyze the $d = 1$ case, which is very easy if treated

with the above techniques, because the Fermi surface contains only two points and therefore there are no longer infinitely many marginal directions in the renormalization group flow.

14. ANALYSIS OF THE RECURSION RELATION. HEURISTIC CONSIDERATIONS ON THE CASE $d=3$

As repeatedly stressed [see observation after (7.10)], the recursion (11.30) for α, ζ, ν involves scalars rather functions on the Fermi surface. A first interesting application of this remark is that the condition of boundedness of the running couplings can be combined with the immediate consequence of (11.30):

$$\begin{aligned} \mathbf{v}_{h-1} &= \Lambda^{-h+1}\mathbf{v}_0 + \sum_{k=h}^{0} \Lambda^{k-h} B_k(\mathbf{v}_k, \ldots, \mathbf{v}_0) \\ &\equiv \Lambda^{-h}\left[\Lambda \mathbf{v}_0 + \sum_{k=h}^{0} \Lambda^k B_k(\mathbf{v}_k, \ldots, \mathbf{v}_0)\right] \end{aligned} \tag{14.1}$$

If we look at the ν component of $\mathbf{v}_{h-1}$, we see that (14.1) becomes

$$\begin{aligned} \nu_{h-1} &= 2^{-h}\left[2\nu_0 + \sum_{k=h}^{0} 2^k B_k^{(\nu)}(\mathbf{v}_k, \ldots, \mathbf{v}_0)\right] \\ &\Rightarrow 2\nu_0 + \sum_{-\infty}^{0} 2^k B_k^{(\nu)}(\mathbf{v}_k, \ldots, \mathbf{v}_0) \equiv 0 \end{aligned} \tag{14.2}$$

where the first relation implies the second if one imposes the boundedness of the form factors:

$$|\mathbf{v}_h| \leqslant M \tag{14.3}$$

Similarly imposing $\alpha_{-\infty} = \zeta_{-\infty}$ we get a scalar equation. Hence, the just mentioned equations are two scalar equations formally fixing the values of ν_0, α_0. If the $(\boldsymbol{\omega}, \boldsymbol{\omega}')$ independence of ν_h, α_h, ζ_h had not been taken into account, it would have appeared that α_0, ν_0 should have been determined by imposing that $\nu_{-\infty}(\boldsymbol{\omega}, \boldsymbol{\omega}') = 0$ and $\zeta_{-\infty}(\boldsymbol{\omega}, \boldsymbol{\omega}') = \alpha_{-\infty}(\boldsymbol{\omega}, \boldsymbol{\omega}')$ and the problem would have looked overdetermined.

Before studying the consequences of the form of the initial interaction on the $\boldsymbol{\omega}$ dependence of the more complicated form factors λ_h, we recall the outcome of the analysis of Sections 11 and 12. Setting $\lambda_h = \lambda$ and $\lambda_{h-1} = \lambda'$ and using a similar notation for the running constants, (11.27) can be written

$$\begin{aligned}
\lambda' &= \lambda + B_{h,1}^{(\geqslant 2)}(\alpha, \zeta, \nu; \lambda) + 2^{\varepsilon h} B_{h,2}^{(\geqslant 0 \geqslant 1)}(\alpha, \zeta, \nu; \lambda) \\
\nu' &= 2\nu + 2^{\varepsilon h} B_{h,3}^{(\geqslant 2)}(\alpha, \zeta, \nu, \lambda) \\
\alpha' &= \alpha + \beta'' \nu^2 + 2^{\varepsilon h} B_{h,4}^{(\geqslant 2)}(\alpha, \zeta, \nu, \lambda) \\
\zeta' &= \zeta - \beta'' \nu^2 + 2^{\varepsilon h} B_{h,5}^{(\geqslant 2)}(\alpha, \zeta, \nu, \lambda)
\end{aligned} \tag{14.4}$$

where we use a notation similar to (11.27), but this time the B functions depend only on the variables on a single scale; they are formal power series with coefficients obeying factorial bounds like (12.4). Thus, we see that all the nonlinear terms referring to α, ζ, ν have in front of them an extra $2^{\varepsilon h}$ with the exception of $\beta'' \nu^2$; hence, they are, order by order, small if the running constants are uniformly bounded.

This can be used to infer that one expects that, fixing conveniently the two free parameters α_0, ν_0, the flow generated by (14.4) has a solution satisfying the two conditions that $\alpha_h - \zeta_h \to 0$, $\nu_h \to 0$; and the convergence to 0 as $h \to -\infty$ of ν_h, $\alpha_h - \alpha_{-\infty}$, $\zeta_h - \zeta_{-\infty}$ can be at a rate of $O(2^h)$:

$$|\nu_h| \leqslant \text{const} \cdot 2^{\varepsilon h}, \qquad |\alpha_h - \alpha_{-\infty}| \leqslant \text{const} \cdot 2^{\varepsilon h}, \qquad |\zeta_h - \zeta_{-\infty}| \leqslant \text{const} \cdot 2^{\varepsilon h} \tag{14.5}$$

Of course, one would like to check the consistency of the assumption (14.3) at least when the beta function recursion is truncated to second order: this is difficult because the recursion relation for λ_h is still too complicated.

The idea to simplify it is to go back to Section 11, where we mentioned the possibility to use a more complicated $\mathscr{L}$ operation, named $\mathscr{L}_C$, keeping the linearity of the operator $\mathscr{L}_C$, thus never breaking the gauge invariance. Then use the fact that the original dependence of the interaction on the fields ψ_x implies, together with the gauge invariance, that the function $\lambda(\omega_1, \omega_2, \omega_3, \omega_4)$ has to be rather special, just as remarked for the $\alpha_h(\omega, \omega')$, $\zeta_h(\omega, \omega')$, $\nu_h(\omega, \omega')$.

A simple calculation shows that the functions λ_h must have the form

$$\lambda_h(\omega_1, \omega_2, \omega_3, \omega_4) = \Lambda_h(\omega_1, \omega_3, \omega_4) - \Lambda_h(\omega_2, \omega_3, \omega_4) \tag{14.6}$$

where the functions Λ_h are rotation invariant and have the symmetry

$$\Lambda(\omega_1, \omega_2, \omega_3) = -\Lambda(\omega_1, \omega_3, \omega_2) \tag{14.7}$$

Hence we can think that the (14.4) is a recursion relation expressing the three scalars α_h, ζ_h, ν_h and the functions $\Lambda_h(\omega_1, \omega_2, \omega_3)$ in terms of the same quantities with higher scale indices. Unfortunately, there seems to be no projection operator which projects a function $\lambda(\omega_1, \omega_2, \omega_3, \omega_4)$ onto

the functions of the form (14.6), defined by means of purely algebraic operations. Hence we cannot make efficient use of the above simplification.

The guide to our attempt to understand the flow of (14.4) has been the conjecture that the λ_h, even in the form (14.6), still contain irrelevant terms which have to be taken out. This is the interpretation of the choice of the new localization operator $\mathscr{L}_C$.

The operator P_C introduced in Section 11 [see (11.5)–(11.7)] acts algebraically on λ_h and therefore one can envisage performing the estimates of Section 10 anew with the localization operator $\mathscr{L}$ replaced by a new localization operator which acts as before on the monomials of degree two. On the monomials of degree four it acts first as in Sections 7 and 10 and successively it changes the localized terms by altering their form factor from λ_h to $P_C\lambda_h$.

Then we easily check that all the contributions to the new B functional in the formula analogous to (9.1) from graphs with more than two λ vertices and at least one loop with three (or more) lines are bounded as demanded by (14.3) and the problem can probably be reduced to the graphs of first and second order in λ (such graphs are the only ones for which one cannot be helped by the extra powers of $2^{\varepsilon h}$ discussed in Section 11). The exchange graph also gives no problem in the estimates, as shown by the bound of δ_2 in (13.27); in fact, $\sum_h J_0(2^{-h}\Delta) \leqslant \text{const}$. But the direct graph definitely gives a contribution contradicting (14.3) if $\lambda_h(\omega_1, \omega_3)$ is just supposed to satisfy (14.3), since the bound (13.26) on δ_1 cannot be improved.

The situation changes if, in addition to (14.3), one supposes that $\lambda_h(\omega_1, \omega_3)$ for $h \to -\infty$ is of the form

$$\lambda_h(\omega_1; \omega_3) = f_h(|2^{-h}(\omega_1 - \omega_3)|) - f_h(|2^{-h}(\omega_1 + \omega_3)|) \tag{14.8}$$

where $f_h(x)$ is a family of functions tending to 0 as $x \to \infty$. Then the integrals in (13.21) and (13.14) can be estimated asymptotically and one gets an explicit form for the leading term of (13.4). Nevertheless, we have not been able to study the recursion equation, not even to second order [i.e., (13.4)], under the hypothesis (14.8).

On a heuristic basis we present a remark which we think is of interest (stressing that from now on the discussion is purely heuristic). The fact that the graphs with more than two four-line vertices do not give problems in deriving bounds like (14.3) should mean that they could actually be completely forgotten in the subtraction procedures. Hence the really important part in the description of the flow of the form factors should be determined by the two basic second-order graphs. One can thus be led to consider seriously the recursion to second order described by the two second-order graphs built with two vertices with four lines.

Consequently we shall try to study the reduced flow generated by the P_C operation [described by (13.14)]. We look for a solution of the flow which asymptotically approaches as $h \to -\infty$ a function $\lambda_h(\boldsymbol{\omega}_1, \boldsymbol{\omega}_3)$ of the form (14.8).

We first want to discard the possibility that λ_h takes the form (14.8) with $f_h \to f$, where f is a regular function not identically zero. It is easy to check, using the estimates on the second-order graphs in Section 13, that the contribution of the direct graph to the flow tends to vanish as $h \to -\infty$, while the exchange graph gives a contribution which keeps f fixed (asymptotically in h), provided f satisfies the fixed-point equation

$$f(x) = f(x/2) - I(x/2)\, f(x/2)^2, \qquad f(\infty) = 0$$

$$I(x) = C \int d\alpha \frac{\alpha^2 \sigma(\alpha)}{c_0^2} \frac{1}{x\beta(p_F/p_0)\sqrt{\alpha}} \int_0^{x\beta(p_F/p_0)\sqrt{\alpha}} y^2 e^{-y^2}\, dy \tag{14.9}$$

$$C \equiv (2\pi)^{-2} p_F^{d-1} \beta^{-1}$$

where $\sigma(\alpha)$ and c_0 have been defined in (13.7) and (13.8); the relations (13.21), (13.24), and (13.25) have also been used.

The relations in (14.9) can be rewritten in dimensionless form by introducing

$$\bar{I}(x) = \int \frac{\alpha^2 \sigma(\alpha)\, d\alpha}{c_0^2} \frac{1}{x\sqrt{\alpha}} \int_0^{x\sqrt{\alpha}} y^2 e^{-y^2}\, dy$$

$$f(x) = gC^{-1} \bar{f}(\beta(p_F/p_0)x) \tag{14.10}$$

$$g \equiv Cf(0)$$

We thus find that the relations in (14.9) become

$$\begin{aligned} &\bar{f}(x) = \bar{f}(x/2) - g\bar{I}(x/2)\, \bar{f}(x/2)^2 \\ &\bar{f}(0) = 1 \end{aligned} \tag{14.11}$$

with the further condition $\bar{f}(\infty) = 0$. It is interesting to remark that the above equation for $\bar{f}$ admits a family of entire solutions. If one imposes the condition $\bar{f}(\infty) = 0$, one finds that the only regular solution is $\bar{f} = 0$.

Hence we see that the exchange graph contribution to the beta function seem unable to keep λ_h away from zero and at the same time a (regular) function of $2^{-h}(\boldsymbol{\omega}_1 - \boldsymbol{\omega}_3)$. On the other hand, one expects that if λ_h does not go to zero, it has to become a function of $2^{-h}(\boldsymbol{\omega}_1 - \boldsymbol{\omega}_3)$; see

below. It is therefore very natural to think that the flow will asymptotically go to a limit regime in which

$$\lambda_h(\boldsymbol{\omega}_1;\boldsymbol{\omega}_3) \xrightarrow[h\to-\infty]{} 0$$
$$\nu_h \to 0, \qquad \alpha_h \to \zeta_{-\infty}, \qquad \zeta_h \to \zeta_{-\infty} \tag{14.12}$$

and the Fermi surface would exist and be normal.

The mechanism whereby $\lambda_h(\boldsymbol{\omega}_1, \boldsymbol{\omega}_3)$ cannot stay smooth in $\boldsymbol{\omega}_1$, $\boldsymbol{\omega}_3$ for too many values of h could be that the exchange graph would be very small, as long as $\boldsymbol{\omega}_1$, $\boldsymbol{\omega}_3$ are fixed (asymptotically it vanishes exponentially, as seen in Section 13) and the flow is determined by the direct graph. The latter, however, acts trivially on the spherical harmonic expansion of λ_h: see (13.18). And if the interaction is repulsive, we see that the $\lambda_l(h)$ component of λ_h evolves, at fixed l and large h, essentially as

$$\lambda_l(h-1) = \lambda_l(h) - \beta\lambda_l(h)^2$$

i.e.,

$$\lambda_l(h) \approx \frac{\lambda_l(0)}{1-\beta\lambda_l(0)h} \tag{14.13}$$

where $\beta > 0$ is $g(0, -\infty)$ with the notations of (13.18), and [by (5.22), (5.23)] $\lambda_h(0) \geqslant 0$.

Hence the low-angular-momentum components are depressed as h grows, provided one understands why they stay nonnegative, and their relative importance diminishes and λ_h looks more and more irregular as a function of $\boldsymbol{\omega}_1$, $\boldsymbol{\omega}_3$, so that the (14.12) seems at least an interesting possibility.

From the above argument we also see that the sign of the potential is relevant and the whole mechanism can only work if the interaction is repulsive.

We refrain from elaborating on the theme, as the discussion would be based on a too strongly conjectural basis.

However, we formulate in a precise fashion our basic conjecture: the map (13.4) acting on functions $\lambda_h(\boldsymbol{\omega}_1, \boldsymbol{\omega}_3)$ with a Legendre expansion, like the first of (13.17), with nonnegative coefficients, is such that the iterates of the map behave as (14.8) with λ_h satisfying (14.12).

15. ONE-DIMENSIONAL MODELS. ANOMALOUS SCALING

In the $d = 1$ case the calculations are simple: the (13.14) still hold, but this time the integrals are in fact sums over two possibilities and they can be immediately computed.

In this section we write $x=(t, \mathbf{x})$, $k=(k_0, \mathbf{k})$ and the quasiparticle momenta will be denoted $\omega=\pm 1$ or, when we want to stress that ω can be thought of as the space component of the vector $(0, 0)$, we shall also use the notation $\boldsymbol{\omega}$.

The factors 2^{-2h} become in the d-dimensional cases $2^{(d-1)h}$; hence, they are absent if $d=1$. Furthermore, the symmetry properties of λ imply that in the expression for δ_1 we must have $\omega=-\omega'$, while in δ_2 the antisymmetry forces $\omega=-\omega_1$, $\omega'=-\omega_3$, $\omega_1=-\omega_3$ in the first term and $\omega=\omega_1$, $\omega'=-\omega_3$, $\omega_1=\omega_3$ in the second; the integrals over ω's are averages, i.e., $\int d\omega = 2^{-1}\sum_{\omega=\pm 1}$, and we see that if

$$\beta_0 = G_h(\mathbf{0}, \omega, -\omega) = 4(4\pi)^2 \beta^3 \int \alpha^3 \sigma(\alpha)\, d\alpha$$

it is

$$\delta_1 = -\beta_0 \lambda^2 + O(2^{2h}), \qquad \delta_2 = \beta_0 \lambda^2 + O(2^{2h}) \tag{15.1}$$

where we set, in this section, $\lambda \equiv \lambda(\omega, \omega)$, $\delta_i \equiv \delta_i(\omega, \omega)$—a convenient notation, as there is only one coupling constant λ in terms of which the $\lambda(\omega_1, \omega_3)$ can be expressed. It is in fact

$$\lambda(\omega_1, \omega_3) \equiv \omega_1 \omega_3 \lambda \tag{15.2}$$

A complete calculation, including the nonleading corrections, is also easy and the result is

$$\delta_1 + \delta_2 = [\beta_1 2^{2h} + o(2^{2h})]\, \lambda^2, \qquad \beta_1 < 0 \tag{15.3}$$

The above cancellation of the leading terms in (15.1) is, however, accidental: to remove it, one should investigate the higher-order terms in the beta function. The calculations of third order, although straightforward, are quite delicate and with many cancellations: we only give the results. We find that the beta functional is, to third order in λ,

$$\lambda_{h-1} = \lambda_h (1 + \beta_3 \lambda_k^2 + \cdots) + 2^{2h} (\beta_1 \lambda_h^2 + \cdots) \tag{15.4}$$

where $0 < \beta_3 < \infty$.

The third- and higher-order terms also contain the other running couplings, but (15.4) is already sufficient to infer that the theory is not asymptotically free. This is unfortunate, as in the one-dimensional case it is clear that the model is a kind of Gross–Neveu model and one should be able to apply the work of ref. 19 to deduce that if $|\lambda_h|$, $|\alpha_h|$, $|\nu_h|$, $|\zeta_h| < M$ and M is small enough, then the formal power series describing the beta

functional is convergent and one could have hoped to study it completely via the technique of refs. 19 and 24.

Unfortunately, ref. 19 treats the running couplings flow and obtains analyticity properties without explicitly referring to our beta functional (introduced in refs. 16–18) and we have not really gone through ref. 19 to check that their results imply the analyticity of our beta functions. Hence, strictly speaking, we do not have yet a formal proof of the analyticity of the beta function. We shall come back to this technical problem in a later publication.

One can hope to use the convergence of the beta functional to look for a different type of asymptotic behavior of the running form factors.

In situations of lack of asymptotic freedom it is by now well known that one ought to look for anomalous scaling behavior: this could manifest itself as an *anomalous dimension* when the relevant Schwinger functions decay at infinity with a power $|x|^{2\eta}$ faster than the corresponding free case; or they decay at infinity faster by $c(\log |x|)^{c'}$. In the first case one says that the system has an *anomalous dimension* η (and η can have any sign); in the second case $\eta = 0$ and the anomaly is just in the c, c' *logarithmic corrections.*

Our formalism, so far, has not been developed so that it could allow for anomalous scalings. It is, however, straightforward to do so, and we only describe the variations of the technical details to adapt the work of Sections 6–10 to the anomalous cases.

We follow a procedure learnt from G. Felder in scalar field theories. The fact that $\omega = \pm 1$ is heavily used, however; hence, what follows is typically one dimensional.

Conceptually one introduces a sequence $Z_0, Z_{-1}, \ldots$ of constants. The constants have to be dynamically determined: it emerges from the coming analysis that the possibility of anomalous scaling can work only if the already analyzed method to exhibit normal scaling fails, and vice versa.

One thinks of defining a sequence of fields $\bar{\psi}^{(\leqslant h)}$ which are defined in terms of the Z_h and of the fields $\psi^{(\leqslant h)}$ with propagators defined by (4.5) as

$$\bar{\psi}^{(\leqslant h)} = Z^{-1/2}\psi^{(\leqslant h)}, \qquad Z = Z_h \tag{15.5}$$

and we shall denote the integration over $\bar{\psi}^{(\leqslant h)}$ by $P_Z(d\bar{\psi}^{(\leqslant h)})$ for $Z > 0$.

The recursive definition of the sequence Z_h proceeds as follows. One starts from $Z_0 = 1$ and considers the relevant functional integral:

$$\int P_{Z_0}(d\bar{\psi}^{(\leqslant 0)}) \exp - V^{(0)}(Z_0^{1/2}\bar{\psi}^{(\leqslant 0)}) \tag{15.6}$$

As already done in Section 14, we can use the remarkable *symmetry* of $V^{(h)}$ implying that $V^{(h)}$ depends only on the particle fields:

$$\bar{\psi}^{(\leq h)\pm}_{\mathbf{x},t} \equiv \int e^{\pm ip_F \omega \mathbf{x}} \bar{\psi}^{(\leq h)\pm}_{\mathbf{x},t,\omega} \, d\omega \tag{15.7}$$

and we can think of $P_{Z_h}(d\bar{\psi}^{(\leq h)})$ as a distribution on the fields $\bar{\psi}^{(\leq h)}_x$ or as a distribution on the quasiparticle fields (as we please!), as long as we integrate functions of the fields depending on $\bar{\psi}^{(\leq h)}_{\mathbf{x},t,\omega}$ via the particle fields (15.7).

We use (15.5) and, hence, the possibility of representing $\bar{\psi}^{(\leq h)}$ as

$$Z_h^{-1/2}\psi^{(h)} + (Z_h/Z_{h-1})^{-1/2}\,\bar{\psi}^{(\leq h-1)} \tag{15.8}$$

where $\bar{\psi}^{(\leq h)}$, $\psi^{(h)}$ are either particle or quasiparticle fields. Integrating (15.6) over $\psi^{(0)}$, we reduce the integration over $\bar{\psi}^{(\leq 0)}$ to an integral over $\bar{\psi}^{(\leq -1)}$:

$$\int P_{Z_{-1}}(d\bar{\psi}^{(\leq -1)}) \exp - \bar{V}^{(-1)}(Z_{-1}^{1/2}\bar{\psi}^{(\leq -1)}) \tag{15.9}$$

where the $\bar{V}^{(-1)}$ is not the same as the effective potential introduced in Section 4. If Z_{-1} is known, then $\bar{V}^{(-1)}$ has to be determined so that the (15.9) holds. We shall see that this is possible in many ways (e.g., $Z_{-1}=1$ has already been discussed): we choose here to determine Z_{-1} by imposing that $\bar{V}^{(-1)}$ *does not contain* one of the relevant terms; precisely we impose that $\mathscr{L}\psi^+ \partial_t \psi^-$ has a vanishing coefficient.

Introduce the kernel operators C_h with Fourier transform

$$C_h(k) = \exp\{+[k_0^2 + (\mathbf{k}^2 - k_F^2)^2]\, 2^{-2h} p_0^{-2}/4\} \tag{15.10}$$

operating on the particle fields $\bar{\psi}_x$ or

$$C_h(k,\omega) = \exp\{+[k_0^2 + (\beta\omega\mathbf{k} + \mathbf{k}^2/2m)^2]\, 2^{-2h} p_0^{-2}/4\} \tag{15.11}$$

operating on the quasiparticle fields $\bar{\psi}_{x,\omega}$. We shall see that the above definition of Z_{-1} is possible only if one allows new fields to appear; the new fields are

$$\begin{aligned} &(\partial_t + i\beta\omega\mathscr{D}_\omega)(1-C_h)\,\bar{\psi}^{(\leq h)}_{x,\omega} \\ &\quad \text{or} \quad [\partial_t + (-\partial^2 - p_F^2)/2m](1-C_h)\,\bar{\psi}^{(\leq h)}_x \end{aligned} \tag{15.12}$$

The actual construction is explained in detail below.

Equations (15.10), (15.11) look horribly diverging at ∞, so that one needs to be very careful in studying the action of C_h: one can apply it only

to fields which have an ultraviolet cutoff strong enough to compensate for the divergence of $C_h(k)$ (as, however, will always be the case).

The new fields $(\partial_t + i\beta\omega \cdot \mathscr{D}_\omega)(1 - C_h)\,\bar\psi^{(\le h)}_{x,\omega}$ will be added to the list (7.1) and the recursive construction of the fields $\bar\psi^{(\le h)}$ and of the Z_h proceeds in the same way as in the normal case.

At the end of the first step we have built Z_{-1} and we can proceed to the construction of the $\bar V^{(-2)}$. The calculations can be performed perturbatively by using the cumulant expansion.

Before proceeding, we give an interpretation of the new integration procedure.

We can think of denoting by $P_{Z_h}(d\bar\psi^{(\le h)})$, $P_{Z_h}(d\bar\psi^{(h)})$ the integrations with respective propagators

$$Z_h^{-1} \sum_{k=-\infty}^{h} 2^k g_k, \qquad Z_h^{-1} 2^h g_h \tag{15.13}$$

To simplify the notation, we do not include in the quasiparticle propagators the $\delta(\boldsymbol{\omega} - \boldsymbol{\omega}')$ functions (which in this case are Kronecker deltas) [see (4.5)], which in some sense are part of them: but of course in computing the graphs we always impose equality of the $\boldsymbol{\omega}$'s of two half lines composing an inner line.

We then define the sequence Z_h, $\bar V^{(h)}$ so that the integral (15.6) is given by

$$\int P_{Z_h}(d\bar\psi^{(h-1)}) \int P_{Z_h}(d\bar\psi^{(h)}) \exp[\bar V^{(h)}(Z_h^{1/2}\bar\psi^{(\le h)})] \tag{15.14}$$

for all h and with $\bar V^{(h)}$ not containing the terms $\mathscr{L}\psi^+ \partial_t \psi^-$.

If the effective potentials are defined and the form factors are bounded, we can interpret this as saying that the Schwinger functions with infrared cutoff at $p_0 2^h$ behave as in the free case with a Z_h correction:

$$S_{(\ge h)}(2^{-h}(x-y), \omega) \simeq S^{\text{free}}_{(\ge h)}(2^{-h}(x-y), \omega)/Z_h \qquad \text{as} \quad h \to -\infty, \quad (x-y) = \text{fixed} \tag{15.15}$$

where $\simeq$ means that the logarithms of both sides, divided by h, have the same limit.

In other words, assuming that, to leading order as $h \to \infty$, one has $Z_h = 2^{-2\eta h}$; we see that this means that the system has an *anomalous dimension* η and if, to leading order, $Z_h = c(-h)^{c'}$, then we see that the system has an anomalous c, c' *logarithmic scaling*.

We do not go through the heuristic argument necessary to establish the (15.15) on the asymptotic properties of the Schwinger functions: this is

well known (see ref. 28, Chapter 9). We just mention that, in the presence of an infrared cutoff at frequency h and on scale $p_0^{-1}2^{-h}$, (3.9) is modified to

$$S_{(\geqslant h)}(x-y)=\frac{1}{Z_h}\, g_{(\geqslant h)}(x-y)-\frac{1}{Z_h^2}\int g_{(\geqslant h)}(x-z) \times V_{\text{eff}}^{(h)}(z-z')\, g_{(\geqslant h)}(z'-y)\, dz\, dz' \tag{15.16}$$

Suppose that one could show that

$$V_{\text{eff}}^{(h)}(z, \boldsymbol{\omega}, z', \boldsymbol{\omega}')=Z_h 2^h \nu_h \delta(z-z')+Z_h \delta_h \delta(z-z')(i\beta\boldsymbol{\omega}'\mathscr{D}_{\boldsymbol{\omega}'}) + Z_h w_h'(z-z') \tag{15.17}$$

where ∂_t', $\mathscr{D}_{\boldsymbol{\omega}'}$ are differential operators acting to their right on the z' variables. Then it would follow that, if $|\mathbf{k}| \sim p_0 2^h$

$$\hat{S}_{(\geqslant h)}(k)=\frac{\hat{g}_{(\geqslant h)}(k)}{Z_h}\,[1-2^h\nu_h-(\mathbf{k}^2-p_{\mathrm{F}}^2)\,\delta_h\,\hat{g}_{(\geqslant h)}(k)-w_h'(k)\,\hat{g}_{(\geqslant h)}(k)] \tag{15.18}$$

Hence we can use the obvious modification of Definitions 1 and 2 of Section 5 to define the anomalous Fermi surface (with the same remarks, limitations, and comments).

The theory of the flow of $\bar{V}^{(h)}$ can be done by using the tree expansion. We begin by writing the functional integral (15.6) as a formal integral over the Grassmanian fields $\bar{\psi}^+$, $\bar{\psi}^-$ of the expression

$$\exp[-T_0(Z_0^{1/2}\bar{\psi})-\bar{V}^{(0)}(Z_0^{1/2}\bar{\psi})] \tag{15.19}$$

where $Z_0=1$, $\bar{V}^{(0)}(Z_0^{1/2}\bar{\psi})\equiv V^{(0)}(\bar{\psi})$, and taking C_0 from (15.10), T_0 is defined by

$$T_0(\bar{\psi})\equiv\sum_{\omega}\int \bar{\psi}^+_{x,\omega}(\partial_t+i\beta\boldsymbol{\omega}\mathscr{D}_{\boldsymbol{\omega}})\, C_0 \bar{\psi}^-_{x,\omega}\, dx \tag{15.20}$$

with $\mathscr{D}$ denoting the covariant space derivative.

Since $\exp[-T_0(Z_0^{1/2}\psi)\, d\psi]$ is the integration with respect to a field with propagator with Fourier transform

$$Z_0^{-1}g^{(\leqslant 0)}(k)\equiv Z_0^{-1}C_0^{-1}(k)(-ik_0+\beta k\omega+k^2/2m)^{-1} \equiv Z_0^{-1}g^{(0)}+Z_0^{-1}g^{(<0)} \tag{15.21}$$

we can write the integration over ψ by representing ψ as $(\psi^{(0)}+\bar{\psi})\, Z_0^{-1/2}$ with $\psi^{(0)}$ having propagator $g^{(0)}(k)$ and $\bar{\psi}$ having propagator $g^{(<0)}(k)$; see Section 4.

The integration over $\psi^{(0)}$ leads, via the tree expansion, to

$$\exp[-T_{-1}(Z_0^{1/2}\psi) - V^{(-1)}(Z_0^{1/2}\psi)] \tag{15.22}$$

where $V^{(-1)}$ is defined exactly as in Sections 6, 7, and 9.

Hence the relevant part $\mathscr{L}V^{(-1)}(Z_0^{1/2}\psi)$ should look like

$$\int dx \Bigg[\sum_{\omega,\omega'} Z_0(n\psi^+_{x,\omega}\psi^-_{x,\omega'} + ia\psi^+_{x,\omega}\boldsymbol{\omega}'\mathscr{D}_{\boldsymbol{\omega}'}\psi^-_{x,\omega'} + z\psi^+_{x,\omega}\partial_t\psi^-_{x,\omega'})\, e^{ip_F(\omega-\omega')\mathbf{x}} + Z_0^2 l\psi^+_{x,1}\psi^+_{x,-1}\psi^-_{x,1}\psi^-_{x,-1}\Bigg] \tag{15.23}$$

We see that (in general $z \neq 0$ and) the relevant term involves a nonzero coefficient z for ∂_t. Therefore, according to the prescription to define Z_{-1}, (15.23) will be rewritten

$$\begin{aligned}
&\int dx\, Z_0 z \sum_{\omega,\omega'} \psi^+_{x,\omega}(\partial_t + i\boldsymbol{\omega}\mathscr{D}_{\boldsymbol{\omega}})\, C_{-1}\psi^-_{x,\omega'} e^{ip_F(\omega-\omega')\mathbf{x}} \\
&\quad + \int dx \Bigg\{ \sum_{\omega,\omega'} [Z_0 n\psi^+_{x,\omega}\psi^-_{x,\omega'} + iZ_0(a-z)\,\psi^+_{x,\omega}\beta\boldsymbol{\omega}'\mathscr{D}_{\boldsymbol{\omega}'}\psi^-_{x,\omega'} e^{i(\omega-\omega')p_F\mathbf{x}}] \\
&\quad + Z_0^2 l\psi^+_{x,\omega}\psi^+_{x,-\omega}\psi^-_{x,\omega}\psi^-_{x,-\omega}\Bigg\} + \Bigg[\int dx \sum_{\omega,\omega'} e^{i(\omega-\omega')p_F\mathbf{x}} \\
&\quad \times Z_0 z\psi^+_{x,\omega}(\partial_t + i\beta\boldsymbol{\omega}'\mathscr{D}_{\boldsymbol{\omega}'})(1-C_{-1})\,\psi^-_{x,\omega'}\Bigg]
\end{aligned} \tag{15.24}$$

The exponential of the sum of (15.24) plus the irrelevant terms has to be integrated with respect to the distribution $P_{Z_0}(d\psi^{(\leq -1)})$. We can again make use of the structure of the effective potentials, as in Section 14, to claim that the integral must be a function of the quasiparticle fields which is expressible in terms of the physical fields. If we imagine for a moment that the effective potentials are expressed in either way, then we see that the free distribution P_{Z_0} in the integration of the exponential of (15.24) plus the sum of the irrelevant terms can be regarded as an integration with respect to particle fields or to quasiparticle fields: the value of the integral of the exponential of (15.24) is the same. (This apparently paradoxical property is most clearly understood in the case of Gaussian integrations, where it can be checked explicitly: it is an algebraic consequence of the Wick integration rule and it holds also in our case where the integrals are only defined via series expansions and the Wick rule: it is in some sense analogous to the Ward identities in quantum electrodynamics.)

The latter remark shows that if we temporarily return to the integral of the exponential of (15.24), plus the irrelevant terms, interpreting the $P_{Z_0}(d\bar{\psi}^{(\leqslant -1)})$ as an integral over the particle fields, we can think of the first term of (15.24) as being expressed in terms of the particle fields as

$$\int dx\, Z_0 z \psi_x^+ [\partial_t + (-\partial^2 - p_F^2)/2m]\, C_{-1} \psi_x^- \tag{15.25}$$

where C_h is defined in (15.10). Hence, we can put it together with the similar term in the free integral $P_{Z_0}(d\bar{\psi}^{(\leqslant -1)})$, changing it to $P_{Z_{-1}}(d\bar{\psi}^{(\leqslant -1)})$, where Z_{-1} is defined by $Z_{-1} = Z_0(1+z)$.

Finally, we can again apply the above remark to think of the integration with respect to $P_{Z_{-1}}(d\bar{\psi}^{(\leqslant -1)})$ as over the quasiparticle fields rather than the particle fields. If we define the operator $\bar{V}_L^{(-1)}(Z_{-1}^{1/2}\bar{\psi})$ to be equal to the term in square brackets in (15.24) and the operator $\bar{V}_R^{(-1)}(Z_{-1}^{1/2}\bar{\psi})$ as the difference between $V^{(-1)}(Z_0^{1/2}\bar{\psi})$ and the sum of the first and second terms of (15.24), we see that $\bar{V}^{(-1)}(Z_{-1}^{1/2}\bar{\psi})$ can be written as a sum of the form

$$\begin{aligned}
&\bar{V}_R^{(-1)}(Z_{-1}^{1/2}\bar{\psi}) + \bar{V}_L^{(-1)}(Z_{-1}^{1/2}\bar{\psi}) \\
&\quad = \bar{V}_R^{(-1)}(Z_{-1}^{1/2}\bar{\psi}) \\
&\qquad \times \int dx \left\{ \left[Z_{-1} \sum_{\omega,\omega'} (i\delta_{-1} \psi_{x,\omega}^+ \omega' \mathscr{D}_{\omega'} \psi_{x,\omega'}^- + 2^{-1} v_{-1} \psi_{x,\omega}^+ \psi_{x,\omega'}^-)\, e^{i(\omega-\omega')p_F x} \right.\right. \\
&\qquad \left.\left. + Z_{-1}^2 \lambda_{-1} \psi_{x,1}^+ \psi_{x,-1}^+ \psi_{x,1}^- \psi_{x,-1}^- \right] \right\}
\end{aligned} \tag{15.26}$$

with Z_{-1} defined after (15.25), and $Z_{-1}\delta_{-1} \equiv Z_0(a-z)$. Furthermore, (15.22) becomes

$$\exp[-T_{-1}(Z_{-1}^{1/2}\bar{\psi}) - \bar{V}^{(-1)}(Z_{-1}^{1/2}\bar{\psi})] \tag{15.27}$$

At this point we iterate the procedure and define $\bar{V}^{(h)}(Z_h^{1/2}\psi)$ of the form

$$\begin{aligned}
&\bar{V}^{(h)}(Z_h^{1/2}\psi) = \bar{V}_L^{(h)}(Z_h^{1/2}\psi) + \bar{V}_R^{(h)}(Z_h^{1/2}\psi) \\
&\bar{V}_L^{(h)}(Z_h^{1/2}\psi) = \int dx \left\{ \left[Z_h \sum_{\omega,\omega'} (i\delta_h \psi_{x,\omega}^+ \omega \mathscr{D}_\omega \psi_{x,\omega'}^- \right.\right. \\
&\qquad \left.\left. + 2^h v_h \psi_{x,\omega}^+ \psi_{x,\omega'}^-) \right] e^{i(\omega-\omega')p_F x} + Z_h^2 \lambda_h \psi_{x,1}^+ \psi_{x,-1}^+ \psi_{x,1}^- \psi_{x,-1}^- \right\}
\end{aligned} \tag{15.28}$$

and the above analysis also implies that $\bar{V}_R^{(h)}$ is expressed by a tree expansion exactly identical to the one associated with the localization operator $\mathscr{L}$ in the normal cases of the preceding sections with the (minor) change that some new two-line vertices may be present in the graphs, corresponding to the new fields

$$(\partial_t + i\beta\omega\mathscr{D}_\omega)(1 - C_{h'})\,\psi^-_{x,\omega} \tag{15.29}$$

with $h' \geqslant h$. The operation $\mathscr{L}$ is extended to the two- and four-external-line graphs containing lines of the form (15.29) simply by using (7.7) and computing the result: it turns out that the result of the action of $\mathscr{L}$ on such operators is simply *zero*.

The beta functional defines a map of the form (9.1) which is constructed by the same rules with more graphs, to take into account the new possibilities that arise because of the new terms with the fields (15.29), and with a suitable factor $(Z_h/Z_{h-1})^n$ multiplying each equation (with $n=2$ for the equation associated with λ_{h-1} and $n=1$ in the cases corresponding to δ_{h-1}, ν_{h-1}).

Furthermore, one finds, of course, no recurrence relation for the ζ_h running constant, which by construction is no longer present. The α_h running coupling, too, is no longer present in some sense: it is replaced by a conceptually new constant which we denote with a different name, δ_h; finally, there is a new equation which replaces the one for the ζ_h and determines the value of Z_h/Z_{h-1} in terms of the previous values of the running couplings.

The lines coming from the fields $(\mathscr{D}_t + ip_F\omega\mathscr{D}_\omega)(1 - C_{h'})\,\psi^-$ originating in the vertices of the above type do not cause problems in the analysis of the new beta function, as they necessarily occur as internal lines and the bigness of $C_{h'}$ is compensated by the ultraviolet cutoff in the propagators. It is in fact easy to see that these internal lines just behave as lines of scale h' and, upon summation over h', as irrelevant terms [the reason is that $1 - C_{h'}(k)$ is big for $k > 2^{h'}$, but the fields on which it operates have ultraviolet cutoff precisely at $k \leqslant 2^{h'}$]. Hence, it behaves as a hard line of scale h' with the power counting (in the dimensional estimates corresponding in this case to those of Section 10) of $(\partial_t + i\omega\partial)\,\psi^-$ and contribute essentially only on the scale h' [and therefore no localization operation is necessary to control their contributions on scales $h < h'$, which explains why $\mathscr{L}$ has been extended as described after (15.29)].

An explicit calculation of the lowest orders of the beta functional is an easy repetition of the previous calculations. The only difference is that one should take into account the contributions from the new two-line vertices (which, however, start at fourth order in the λ_h equation and third order in the δ_h equation).

The result is very similar to the normal scaling case (11.31):

$$\begin{aligned}
\lambda_{h-1} &= (Z_h/Z_{h-1})^2 \,[\lambda_h + \lambda_h^3 B_1(\lambda_h) + \delta_h \lambda_h^2 B_2(\lambda_h, \delta_h) \\
&\quad + \nu_h^2 B_3(\lambda_h, \delta_h, \nu_h) + 2^h \bar{R}_1(\lambda_h, \delta_h, \nu_h, 2^h)] \\
\delta_{h-1} &= (Z_h/Z_{h-1})[\delta_h + \lambda_h^2 \delta_h B_4(\lambda_h) + \nu_h^2 B_5(\lambda_h, \delta_h, \nu_h) \\
&\quad + 2^h \bar{R}_2(\lambda_h, \delta_h, \nu_h, 2^h)] \\
\nu_{h-1} &= 2(Z_h/Z_{h-1})[\nu_h + \nu_h \lambda_h^2 B_6(\lambda_h) \\
&\quad + \delta_h \lambda_h^2 B_7(\lambda_h, \delta_h, \nu_h) + 2^h \bar{R}_3(\lambda_h, \delta_h, \nu_h, 2^h)] \\
1 &= (Z_h/Z_{h-1})[1 + \lambda_h^2 B_8(\lambda_h) \\
&\quad + \delta_h \lambda_h^2 B_9(\lambda_h, \delta_h) + \lambda_h^2 \nu_h B_{10}(\lambda_h, \delta_h, \nu_h) \\
&\quad + 2^h \bar{R}_4(\lambda_h, \delta_h, \nu_h, 2^h)]
\end{aligned} \tag{15.30}$$

where we have computed a little more carefully the lowest terms to find out the minimal power to which each running constant is raised; and the functions B_j, $\bar{R}_j$ are analytic in their arguments λ_h, δ_h, ν_h (with a suitably small radius M of convergence). This convergence, for $|\lambda_k|$, $|\nu_k|$, $|\delta_k|$ small enough, should again be a consequence of ref. 19, with the same warning spelled out in the comment following (14.4) and in the Introduction. Furthermore, the B_j can be taken h independent (note, however, that the $\bar{R}_j$, depending explicitly on $t = 2^h$, introduce an h dependence). The $\bar{R}_j$ vanish to second order in λ_h, δ_h, ν_h.

Note that the terms depending only on λ_h are *missing* from the second of (15.30) in the B part: this is essential and it represents one more motivation for introducing the anomalous dimension. They disappear as a consequence of the fact that at each step δ_h is essentially the difference between the old running constants $\alpha_h - \zeta_h$ and the variations of α_h, ζ_h have the same term of $O(\lambda_h^2)$; see the B_5 terms in (11.31). A moment of thought shows that this may mean that the discussion of the qualitative behavior of the iterates of the map (15.30) is rather different from the corresponding normal scaling case.

The lowest orders of the functions B_j are explicitly computable; for instance, $B_1(\lambda) = \beta_3 + \cdots$, or $B_8(\lambda) = \beta_2' + \cdots$, where the coefficients β_3, β_2' are the *same* as those already introduced in the case of the normal scaling discussion [see (15.4)].

It is convenient to eliminate completely the factors Z_h/Z_{h-1} from (15.30), using the last of (15.30) and expanding the denominators in powers series:

$$\begin{aligned}
\lambda_{h-1} &= \lambda_h + \lambda_h^3 G_1(\lambda_h) + \delta_h \lambda_h^2 G_2(\lambda_h, \delta_h) + \nu_h^2 \lambda_h^2 G_3(\lambda_h, \delta_h, \nu_h) \\
&\quad + t_h R_1(\lambda_h, \delta_h, \nu_h, t_h) \\
\delta_{h-1} &= \delta_h + \lambda_h^2 \delta_h G_4(\lambda_h, \delta_h) + \lambda_h^2 \nu_h G_5(\lambda_h, \delta_h, \nu_h) + t_h R_2(\lambda_h, \delta_h, \nu_h, t_h) \\
\nu_{h-1} &= 2\nu_h + \nu_h \lambda_h^2 G_6(\lambda_h, \delta_h, \nu_h, t_h) + \delta_h \lambda_h^2 G_7(\lambda_h, \delta_h, \nu_h, t_h) \\
&\quad + R_3(\lambda_h, \delta_h, \nu_h, t_h) \\
t_{h-1} &= 2^{-1} t_h
\end{aligned} \tag{15.31}$$

having set $t_h \equiv 2^{+h}$: the functions G_j, R_j are smooth in the (trivial) parameter t_h to any order and are analytic in their arguments λ_h, δ_h, ν_h for small values, uniformly in t_h. The R_j vanish to second order in the λ, δ, ν variables.

Clearly the first question is whether $\beta_3 \gtrless \beta_2'$. If $\beta_3 < 2\beta_2'$, we see that the first and the fourth of (15.30) have interesting consequences. If by fixing suitably the initial δ_0 (i.e., the initial α_0) and ν_0, the constants δ_h and ν_h approach 0 as $h \to -\infty$, then $\lambda_h \to 0$ as $h \to -\infty$ and this happens at the rate $O(1/|h|^{1/2})$; hence

$$Z_h \approx \exp \sum_{k=h}^{0} \log(1 + \beta_2' \lambda_k^2) \approx c\,|h| \tag{15.32}$$

and a more careful analysis shows that $c = \beta_2' \lambda_0^2 [1 + o(\lambda_0)]$, and we see that we have, in this case, logarithmically anomalous scaling with exponent $c' = 1$ and c proportional to λ_0^2.

Let $\beta_3 > 2\beta_2'$ and consider the analytic function $B(\lambda) = \lambda^2 G_1(\lambda)$. Then we look for a nontrivial solution λ^* of the equation

$$\lambda = \lambda[1 + B(\lambda)] \tag{15.33}$$

Clearly the existence of such a solution would be easy if the size of the ratio $\beta_3/2\beta_2'$ was very big and if β_3^{-1} was very small compared to the radius of convergence of the series for B_1, B_8: and in this case the point λ^* would be an attractor for the map $\lambda' = \lambda[1 + B(\lambda)]$. Therefore it could be possible to deduce from (15.30) that if λ_0 was small enough, then ν_0, δ_0 could be so chosen that the complete flow behaved as

$$\lambda_h \to \lambda^*, \qquad \delta_h, \nu_h \to 0, \qquad Z_h \approx 2^{-2\eta h} \equiv [1 + B_8(\lambda^*)]^{-h} > 1 \tag{15.34}$$

and therefore we would have positive anomalous dimension independent of the actual value of the initial λ_0. Heuristically this should mean that the singularity at $|\mathbf{k}| = p_F$ of the Fourier transform of the Schwinger function changes nature from the discontinuity of (0.2) to a singularity $|\mathbf{k}^2 - p_F^2|^{2\eta}\,\mathrm{sign}(p_F - |\mathbf{k}|)$.

The existence of the nontrivial fixed point could be checked if the above inequalities among the constants β_2', β_3 and the radius of convergence were strong enough: one could envisage trying a computer-assisted proof of this fact. One is, however, a little hesitant at starting this program, as there is no guarantee that it will work, since the inequalities we hope for are, in any case, just sufficient conditions for the existence of the anomalous Fermi surface.

Fortunately, this extra work, and a rigorous analysis of the above possibilities, seems not to be necessary at all. There is a third possibility, not yet examined, namely $\beta_3 = 2\beta_2'!$, and one may even envisage that in (15.31)

$$G_1 \equiv 0 \tag{15.35}$$

i.e., the *leading term* in (15.31) vanishes, and one is left only with the corrections to scaling containing the asymptotically vanishing factor $t_h \equiv 2^h$ and terms containing v_h or δ_h as factors.

This is a very interesting possibility: philosophically, the best, because it permits one to have an anomaly which varies continuously with the strength of the interaction at least for small interaction. In such a case the flow will be entirely determined by the terms proportional to t_h and hence of size $O(2^h)$, and it will be very trivial. Fixing conveniently v_0, δ_0, we would have a flow in which

$$\begin{aligned} &\lambda_h \xrightarrow[h \to -\infty]{} \lambda_{-\infty}(\lambda_0), \qquad && \frac{Z_h}{Z_{h-1}} \xrightarrow[h \to -\infty]{} \frac{1}{\kappa(\lambda_0)} \\ &v_h \xrightarrow[h \to -\infty]{} 0, && \delta_h \xrightarrow[h \to -\infty]{} 0 \end{aligned} \tag{15.36}$$

with $\lambda_{-\infty}$, $\zeta_{-\infty}$ analytic near $\lambda_0 = 0$ and with $\lambda_{-\infty}$ divisible by λ_0 and $\log \kappa(\lambda_0)$ positive near 0 and divisible by λ_0^2.

Since the series for the G_j are convergent, the above is the only way one could have a flow of running couplings implying anomalies of the type (15.36) under the additional assumption that λ_h, δ_h, v_h never get out of a circle of radius $O(\lambda_0)$. This is an anomaly $2\eta = \log_2 \kappa(\lambda_0) = O(\lambda_0^2) > 0$, which is very different from the previously considered ones, which were independent of λ_0.

Of course, one could envisage intermediate cases in which the functions G_j vanish up to a finite order only. This would not change the situation, as it is easy to see that if they vanish to order n, then the anomaly is logarithmic with $c' = 1 - 2/n$, hence again independent of λ_0 and different from (15.36).

Therefore, a way to prove that (15.35) holds is simply to show the existence of a model in which there is anomalous dimension analytic in λ_0 at 0 and in which the flow of the constants δ_h and ν_h is trivial.

It is not even necessary that the model be in the class introduced in Section 1 and considered so far; it is sufficient that it can be studied via the renormalization group and that it is described by a beta function with the same G_j as above.

Such models do indeed exist and can be taken to be the Luttinger model or its slight variation introduced by Mattis and Lieb[25–27]: the exact solution of such models by Mattis and Lieb[26,27] shows that it has anomalous dimension $\eta = O(\lambda_0^2)$.

We shall choose here, as a reference model, the above Mattis–Lieb variation of the Luttinger model and not the Luttinger model itself (which, in many respects would be equally good) because (a) the variation has been adapted to spinning models,[30] and later we want to make some comments on the extension of our work to such cases, and (b) the bosonic representation (15.46) used below works for the Mattis–Lieb model.

On the other hand, the Mattis–Lieb variation of the Luttinger model corresponds in our language to a model confined in an interval $[0, L]$ with periodic boundary conditions and with propagator

$$g(t, \mathbf{x}; \boldsymbol{\omega}) = \frac{1}{(2\pi)^2} \int dk_0 \, d\mathbf{k} \, \frac{e^{i(k_0 t + \mathbf{k}\mathbf{x})}}{-ik_0 + \beta \boldsymbol{\omega}\mathbf{k}} \tag{15.37}$$

with an interaction somewhat artificial because it cannot be written in terms of the particle fields $\psi_{t,\mathbf{x}}$ but only in terms of the fields $\psi_{(t,\mathbf{x}),\omega}$, which are the quasiparticle fields (thus deprived of a direct physical interpretation). Nevertheless, it is an interaction acceptable by our formalism, which works always in the quasiparticle language. The interaction is defined to be

$$\begin{aligned} &\int dt \, d\mathbf{x} \, d\mathbf{y} \, \lambda_0(\mathbf{x} - \mathbf{y}) \, \delta(t - t') \\ &\quad \times \left(\sum_{\omega} \psi^{+}_{(t,\mathbf{x}),\omega} \psi^{-}_{(t,\mathbf{x}),\omega} \right) \left(\sum_{\omega} \psi^{+}_{(t',\mathbf{y}),\omega} \psi^{-}_{(t',\mathbf{y}),\omega} \right) \\ &\quad + \sum_{\omega} \int \nu \psi^{+}_{x,\omega} \psi^{-}_{x,\omega} \, dx + \int \sigma \, dx + \Delta E \end{aligned} \tag{15.38}$$

where ν, σ are suitable (possibly divergent) constants, and ΔE is a trivial extra term [described below; see (15.40)].

We shall fix our model to be (15.38) *deprived* of the extra term ΔE just because some formulas are neater (but we stress that this modification

changes the Mattis–Lieb model in a trivial way: the energy levels are shifted and the Schwinger functions do not change).

The reason for the introduction of the ν, σ constants has its root in the fact that (15.37), with a *linear dispersion relation*, gives rise to theories with an ultraviolet problem.

The ν, σ can be determined by the introduction of a sharp ultraviolet cutoff, say at $2^U p_0$ with p_0^{-1} = range of the potential λ_0; and by imposing that the field theory is well defined on scale 0 and on this scale it is described (uniformly in U) by an effective potential $V^{(0)}$ which has short range in the same sense in which the effective potentials $V^{(h)}$ in the previous theories have short range for $h < 0$.

In particular, it will be possible to identify a relevant part and an irrelevant part of $V^{(0)}$ and, setting $Z_0 = 1$, proceed to study the flow of $V^{(h)}$ for $h < 0$ in the same way as in the real model: it would be simpler, as one can now simply use only the quasiparticle fields without needing the symmetry allowing us to switch from particle fields to quasiparticle fields when desired [this is quite fortunate, as the symmetry, in fact, is not present in this case, being already broken in the initial interaction (15.38)].

It should be noted that ν, σ do depend upon the ultraviolet regularization chosen to give a meaning to the initial Hamiltonian. Such ultraviolet regularization is not explicitly mentioned in refs. 25–27; it appears, however, that the authors proceed as if one had a sharp cutoff at frequency U on the space momenta (i.e., there are no particles or holes with momentum $|\mathbf{k}| \geqslant p_0 2^U$).

In this case, however, since the model is soluble, one can in fact even compute explicit expressions for the counterterms. It is easily found (by suitably interpreting ref. 26) that the correct choice is

$$\nu = -4(2^U p_0 + p_F)\,\hat{\lambda}(0)/2\pi, \qquad \sigma = (2^U p_0 + p_F)^2\,\hat{\lambda}(0)/\pi^2 \tag{15.39}$$

This result stems from the fact that one can check that the interaction (15.38) in the Grassmanian fields corresponds to a Hamiltonian of interaction equal to the operator H' in (4.6) of ref. 27 plus (using here the notations of ref. 27) a (finite) correction:

$$\Delta E = \frac{\hat{\lambda}(0)}{L}\left[\sum_{k>0}\left(a^*_{k,1}a_{k,1} - a_{-k,1}a^*_{-k,1} + a^*_{-k,2}a_{-k,2} - a_{k,2}a^*_{k,2}\right)\right]^2 \tag{15.40}$$

Alternatively, one can see that (15.39) inserted into (15.38) can be simply rewritten as (15.38) itself without the σ, ν terms, provided that in the fourth-degree interaction term one replaces the $\psi^+\psi^-$ products by their Wick products: this prescription seems better, as it is formally regularization independent. If we decided to use a regularization with rotational

symmetry (for $\pi/2$ rotations) in the space of $(k_0, \mathbf{k})$, then the Wick product would coincide with the ordinary product because the propagator so regularized would vanish at zero distance, by symmetry. In this case the parameters v, σ would vanish, i.e., they would be very different from those in (15.39).

The extra term (15.40) in the interaction is an operator which commutes with the rest of the Hamiltonian: hence our variation of the Mattis–Lieb version of the Luttinger model is a trivial one and it can be solved by exactly the same method and leading to the same ground state and to the same pair Schwinger function [the excited levels are trivially related to those of the model without the extra term (15.40)].

For purposes of comparison with refs. 26 and 27, one considers operators in Fock space, making the following identification of the operators $\psi_1^{\pm}$, $\psi_2^{\pm}$ with our $\psi_\omega^{\pm}$:

$$\psi_1^{\pm}(\mathbf{x}) \leftrightarrow e^{\pm ip_F x} \psi^{\pm}_{(\mathbf{x},0),+}, \qquad \psi_2^{\pm}(\mathbf{x}) \leftrightarrow e^{\mp ip_F x} \psi^{\pm}_{(\mathbf{x},0),-} \tag{15.41}$$

Note that in refs. 26 and 27 only the fields $\psi_\alpha^{\pm}(\mathbf{x}, 0) \equiv \psi_\alpha^{\pm}(\mathbf{x})$ are considered; but consistently with Section 2 here, one could introduce $\psi_\alpha^{\pm}(\mathbf{x}, t)$ as

$$\begin{aligned} \psi_\alpha^{\pm}(\mathbf{x}, t) &= e^{tT_0} \psi_\alpha^{\pm}(\mathbf{x})\, e^{-tT_0} \\ T_0 &= \int_0^L d\mathbf{x}\, \{[\psi_1^+(\mathbf{x})\, \partial\psi_1^- - \psi_2^+(\mathbf{x})\, \partial\psi_2^-] \\ &\quad - p_F[\psi_1^+(\mathbf{x})\, \psi_1^-(\mathbf{x}) + \psi_2^+(\mathbf{x})\, \psi_2^-(\mathbf{x})]\} \end{aligned} \tag{15.42}$$

so that, in general,

$$\psi_1^{\pm}(\mathbf{x}, t) = e^{\pm ip_F x} \psi^{\pm}_{(\mathbf{x},t),+}, \qquad \psi_2^{\pm}(\mathbf{x}, t) = e^{\mp ip_F x} \psi^{\pm}_{(\mathbf{x},t),-} \tag{15.43}$$

Proceeding as in Section 3, one can put the problem in the language of functional integration with respect to Grassmanian fields with propagator (15.37).

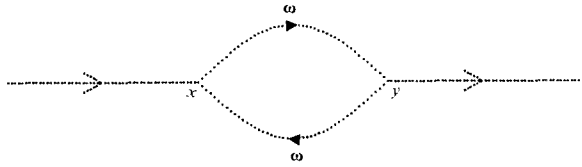
In this model the ultraviolet cutoff is lowered from U to 0 by using methods of superrenormalizable field theory. The lowering of the cutoff is a necessary step because our methods work only if the range of the interaction is the same as that of the ultraviolet cutoff. In the real model this is a minor problem (as mentioned in the Introduction), but in the case of the Mattis–Lieb model it is more serious.

The reason is that the free propagator behaves, in momentum space, as $1/|k|$ when either k_0 or $\mathbf{k}$ goes to ∞: while in the real model the behavior as $k \to \infty$ is as $1/\mathbf{k}^2$.

For heuristic purposes one can begin by studying a formal perturbation theory expression for the effective potential, on scale p_0^{-1}, $V^{(0)}$: and in fact this suggests what to do to go beyond the formal level.

One starts by integrating the high-frequency components of the field (i.e., the frequencies higher than p_0) directly in one step, without using any scale decomposition. We do not use the graph elements of Section 8, (8.1), (8.2); if we adopt the convention of writing the interaction in Wick ordered form, there is only one graph element that matters, namely the first of (2.6).

If one examines the perturbation theory formulas for the effective potential, one realizes that ultraviolet divergences can only be present in subgraphs of the form



(15.44)

However, the (logarithmic) divergence is canceled by the summation over ω and by the symmetry between space and time, so that there is no divergence and the momentum dependence (at large momenta) of the subgraph (15.44) is $\hat{\lambda}(\mathbf{k})^2 \sigma(k)$ with $\sigma(k)$ bounded [instead of the *a priori* $\sigma(k) \simeq \log |k|$]. Hence, no divergence really arises—as expected from the above property of the exact solution, which implies that the problem is well defined as soon as one writes the interaction in Wick ordered form. It also appears that once $\sigma(k)$ is bounded, the sign of the initial interaction potential does not matter, as long as the strength is small enough.

The above argument shows that there is no problem at the level of perturbation theory: if, however, one wants a fully nonperturbative analysis, one has to discuss the mechanism which permits us to put bounds on the kernels defining the effective potential $V^{(0)}$. We can do this under the extra assumption that the initial potential is positive definite: however, we think that this is a limitation due only to the technique that we develop. And if one just wants a perturbative theory without control of the convergence, this limitation is not necessary, just as it also emerges from the study of the exact solution in ref. 26.

Assuming $\hat{\lambda} \geqslant 0$, we simply use the well-known trick of the introduction of an auxiliary boson field $\varphi_{x,t}$ with propagator F with Fourier transform

$$\hat{F}(k_0, \mathbf{k}) \equiv \hat{\lambda}(\mathbf{k})/\hat{\lambda}(0) \tag{15.45}$$

and write the interaction

$$\sum_{\omega} \hat{\lambda}(0)^{1/2} \int \varphi_x : \psi^{+}_{x\omega} \psi^{-}_{x\omega} : dx \tag{15.46}$$

which, upon integration over φ, manifestly generates the Mattis–Lieb model.

We study the model (15.46) by decomposing the propagators into scales by writing

$$\begin{aligned}\hat F(k_0,\mathbf{k}) &\equiv \sum_{n=1}^{\infty} \hat F_n(k)\\ &\equiv \{\exp[-(k_0^2+\mathbf{k}^2)\,p_0^{-2}/4]\}\,\hat\lambda(\mathbf{k}) + \sum_{n=2}^{\infty}\{\exp[-(k_0^2+\mathbf{k}^2)\,p_0^{-2}2^{-2n}]\\ &\quad - \exp[-4(k_0^2+\mathbf{k}^2)\,p_0^{-2}2^{-2n}]\}\,\hat\lambda(\mathbf{k})\\ \hat g(k_0,\mathbf{k}) &\equiv g_{(\leqslant 0)} + \sum_{n=1}^{\infty}\hat g_0(2^n k)\end{aligned} \tag{15.47}$$

where we note that the decomposition of g has exact scaling properties (because it corresponds to the propagator of the Mattis–Lieb model and hence it is different from the one considered in Section 4 valid for the analysis of the infrared problem in the real model: note, however, that g_0 is nothing but the function appearing in the leading term of the infrared propagator of the real model).

We analyze next the size of $F_n(x)$ and we easily find that it is bounded uniformly in n by const $\cdot\, 2^n$. The size of $g_n(x)$ admits the same bound. Hence we can perform the usual dimensional analysis of the size of the interaction at high frequency.[17] Replacing φ by const $\cdot\, 2^{n/2}$, and ψ by the same quantity, and restricting the integral to a box of size $2^{-2n}p_0^{-2}$, we see that the size of the interaction at large scale (i.e., large n) is const $\cdot\, \hat\lambda(0)\, 2^{-n/2}$. Hence, the model is still asymptotically free in its ultraviolet part and even superrenormalizable. It can have divergences up to order 4 in perturbation theory: but the interaction structure is such that only even orders in the coupling can be present and furthermore the expansion parameter squared is the size of the potential. Hence, the only divergent graphs are to be looked for among the terms of second order in the potential λ_0 [which are already of fourth order in the sense of (15.46)].

Therefore we can apply known expansion methods of constructive field theory for renormalizable theories; see refs. 24 and 32 for the most recent developments.

Ours can be regarded as a two-dimensional Yukawa theory with a boson propagator less singular than usual; furthermore, the general techniques of ref. 32 can be applied to our case.

In this way one shows that the effective potential on scale 0 is a short-range potential with many-body components (i.e., terms containing any number of $\psi^{\pm}$ fields) which become very small as the number of bodies

increases. Since the field φ has no components on scales lower than p_0, the effective potential $V^{(0)}$ can be taken as the starting point of our infrared analysis. The novelty is simply that the initial number of irrelevant terms is not finite, because of the fact that the effective potential contains components with arbitrarily many fields.

The latter is not a big problem because even if we had only finitely many components in $V^{(0)}$, we would immediately generate infinitely many new ones after the integration of the first infrared component of the field. Thus, we only have to check that $V^{(0)}$ obeys bounds on the kernels of the n field terms which are no worse than those that $V^{(-1)}$ would obey in the case it was generated by an initial interaction with only finitely many terms. This is precisely the type of bounds that are provided by the known methods[24,32] for the kernels of the effective potential on scale 0 generated by a superrenormalizable, or even just asymptotically free, interaction. One also finds that in such cases the kernels for $V^{(0)}$ admit an asymptotic expansion in powers of the initial coupling, which in our case might even be convergent.

From this point on one continues as in the realistic case treated above. The fact that the propagator (15.37) is *on all scales* our scaling propagator makes the theory of the beta function even simpler because the parts proportional to $t_h = 2^h$, which were corrections to scaling, are absent and the functions G_j which only depend on the scaling propagator *are the same as ours.*

At this point we have to make an assumption that we have not been able to deduce from the exact solution:

Assumption. The running couplings in the Mattis–Lieb model stay smaller than $C\lambda_0$ for some constant C for all values of h.

From the above argument, we know that they start being as small as desired and from the exact solution [see (15.48) below], we know that they end up being as small as desired; unfortunately, one cannot exclude yet that while going down in frequency they become large, leaving the perturbative regime and returning into it. This event, which we consider unlikely, can perhaps be excluded by a more careful analysis of the exact solution.

Furthermore, we consider the appendix of ref. 26 and note that our $\hat{\lambda}(0)$ is half the quantity denoted $\lambda v(0)$ in ref. 26. The results of refs. 26 and 27 prove that the model shows anomalous scaling with η given by

$$2\eta = \{[1 + 2\hat{\lambda}_0(0)/\pi]^{1/2} + [1 + 2\hat{\lambda}_0(0)/\pi]^{-1/2} - 2\}/2 = \hat{\lambda}_0(0)^2/2\pi^2 + \cdots \tag{15.48}$$

where $\hat{\lambda}_0$ is the Fourier transform of λ_0 and, in our notation, coincides [see (15.38)] with the quantity $2\lambda v(0)$ of ref. 27.

Note also that if we start from an initial interaction containing a non-zero δ coefficient and a nonzero ν coefficient, the model is of course still soluble, because this amounts simply to changing the value of p_F and of the coefficient of $\partial\psi$, thus affecting in a trivial (analytic) way the end result (15.48).

This result holds for any choice of λ_0 small enough, a property incompatible with a flow described by (15.31) unless $G_1 \equiv 0$, if one accepts the assumption above.

The realistic model and the Luttinger models have the same G_1 functions; hence $G_1 \equiv 0$ for both.

If $G_1 = 0$, the flow (15.31) is easy to study. We look for a solution in which $\delta_h = \gamma^h \bar{\delta}_h$, $\nu_h = \gamma^h \bar{\nu}_h$, $t_h = 2^h$, and $\bar{\delta}_h$, $\bar{\nu}_h$ tend to zero. The recursion becomes, for any γ,

$$\begin{aligned}
\lambda_{h-1} &= \lambda_h + \gamma^h[\bar{\delta}_h \lambda_h^2 G_2(\lambda_h, \bar{\delta}_h) + \bar{\nu}_h^2 \lambda_h^2 \gamma^h G_3(\lambda_h, \bar{\delta}_h, \bar{\nu}_h)] \\
&\quad + t_h R_1(\lambda_h, \bar{\delta}_h, \bar{\nu}_h, t_h) \\
\delta_{h-1} &= \gamma \delta_h + \lambda_h^2 \bar{\delta}_h \gamma G_4(\lambda_h, \bar{\delta}_h) + \lambda_h^2 \bar{\nu}_h \gamma G_5(\lambda_h, \bar{\delta}_h, \bar{\nu}_h) \\
&\quad + t_h \gamma^{-h} \gamma R_2(\lambda_h, \bar{\delta}_h, \bar{\nu}_h, t_h) \qquad (15.49) \\
\bar{\nu}_{h-1} &= 2\gamma \bar{\nu}_h + \bar{\nu}_h \lambda_h^2 \gamma G_6(\lambda_h, \bar{\delta}_h, \bar{\nu}_h, t_h) + \bar{\delta}_h \lambda_h^2 \gamma G_7(\lambda_h, \bar{\delta}_h, \bar{\nu}_h, t_h) \\
&\quad + t_h \gamma^{-h} \gamma R_3(\lambda_h, \bar{\delta}_h, \bar{\nu}_h, t_h) \\
t_{h-1} &= 2^{-1} t_h
\end{aligned}$$

Taking γ between 1 and 2 and fixing λ_0 small enough, it is clear that one can find δ_0, ν_0 so that $\bar{\delta}_h \to 0$, $\bar{\nu}_h \to 0$ as fast as $O[(2/\gamma)^h]$, so that δ_h, ν_h tend to zero as $O(2^h)$ while $\lambda_h \to \lambda_{-\infty}$.

If there is a flow on (λ, δ, ν) satisfying (15.31) and staying bounded of $O(\lambda_0)$, it is clear that $G_1 \equiv 0$ is the only possibility.

It follows immediately that, if the above assumption holds, all the short-range models in one dimension have an anomalous Fermi surface at small coupling: the anomaly is correctly caught by the Luttinger model. Probably at large coupling the anomaly remains and follows the pattern predicted by the Mattis–Lieb exact solution of the Luttinger model; see ref. 26. The only exception is the set of models for which $\lambda_{-\infty}$ vanishes: such cases are *not generic*, as the $\lambda_\infty = \lambda_0 + \beta\lambda_0^2 + \cdots$ depend analytically on λ_0; the value λ_0 depends smoothly on the initial interaction and to first order it is $\hat{\lambda}(0) - \hat{\lambda}(2p_F)$, as can be checked immediately by an elementary perturbation calculation. This concludes our analysis of the one-dimensional cases.

Note that we do not have to require, in the realistic case, that the interaction be positive definite. Positive definiteness of λ_0 is used above only as an intermediate step to conclude, via the theory of the Mattis–Lieb model, that $G_1 = 0$. If, however, in the realistic model the interaction is positive definite, then we can use the above method of introducing an auxiliary boson field to solve the ultraviolet problem mentioned at the beginning of the Introduction. Hence the theory is complete in this case. As mentioned in the Introduction, we do not think that this is a serious problem even in the attractive case, for small enough interaction. Physically the only thing that could conceivably go wrong is the extensivity of the lower bound on the energy levels, i.e., the thermodynamic stability: but if $d = 1$, the indeterminacy principle allows us to consider a negative interaction without destroying stability, provided it is not too large.

The degeneracy of the second order in (15.2) (which tends to 0 as $h \to -\infty$) does not occur if there are more degrees of freedom: if we suppose that our fermions, and the quasiparticles as well, carry a *spin* described by an extra label σ attached to the fields, the formalism that we have described applies unchanged, with the obvious addition of the extra labels. As an illustration, we present a particularly simple spinning model.

The model has a potential

$$\sum_{\sigma,\sigma' = \pm 1} \int \lambda_0(\mathbf{x} - \mathbf{y}) :\psi^+_{x\sigma} \psi^+_{y\sigma'} \psi^-_{y\sigma} \psi^-_{x\sigma'}: d\mathbf{x}\, d\mathbf{y}\, dt \tag{15.50}$$

which represents a spin-symmetric interaction. Its relevant part is of the form

$$\sum_{\sigma,\sigma' = \pm 1} \int dx \prod_{i=1}^{4} d\omega_i\, e^{ip_F(\omega_1 + \omega_2 - \omega_3 - \omega_4)\mathbf{x}} \lambda_0(\omega_1, \omega_2, \omega_3, \omega_4) \times \psi^+_{x,\sigma,\omega_1} \psi^+_{x,\sigma',\omega_2} \psi^-_{x,\sigma',\omega_3} \psi^-_{x,\sigma,\omega_4} \tag{15.51}$$

where, if $\lambda(\omega - \omega') = \hat{\lambda}_0(p_F(\omega - \omega'))$ denotes the Fourier transform of λ_0 evaluated at the difference between a pair of Fermi momenta, one has

$$\lambda_0(\omega_1, \omega_2, \omega_3, \omega_4) = [\lambda(\omega_1 - \omega_4) + \lambda(\omega_2 - \omega_3)]/2 \tag{15.52}$$

The coupling $\lambda_0(\omega_1, \omega_2, \omega_3, \omega_4)$ in (15.51) has some symmetries, which are preserved by the (normal scaling) beta function, that is,

$$\begin{aligned} \lambda_0(\omega_1, \omega_2, \omega_3, \omega_4) &= \lambda_0(-\omega_1, -\omega_2, -\omega_3, -\omega_4) \\ &= \lambda_0(\omega_2, \omega_1, \omega_4, \omega_3) = \lambda_0(\omega_3, \omega_4, \omega_1, \omega_2) \end{aligned} \tag{15.53}$$

This reduces to five the number of independent parameters; for example,

$$y_1 = \lambda_0(1, 1, 1, 1), \qquad y_2 = \lambda_0(1, 1, 1, -1), \qquad y_3 = \lambda_0(1, 1, -1, -1)$$
$$y_4 = \lambda_0(1, -1, 1, -1), \quad y_5 = \lambda_0(1, -1, -1, 1) \tag{15.54}$$

The evaluation of the second-order beta function is straightforward but laborious. One finds, up to terms vanishing exponentially in $h \to -\infty$,

$$y_1' = y_1, \qquad y_2' = y_2 - \beta y_2(2y_4 - y_5)$$
$$y_3' = y_3 - 2\beta y_3(y_4 - 2y_5), \qquad y_4' = y_4 - 4\beta y_4^2 \tag{15.55}$$
$$y_5' = y_5 - 2\beta y_4^2$$

where β is a positive constant. It is easy to see that in this dynamical system no trajectory approaches the origin: i.e., the introduction of the spin does not change the nonasymptotically free nature of the model.

The above remarks show that we cannot make a connection with the theory of the model of Gross and Neveu as treated in ref. 19 because we do not have asymptotic freedom to second order. Nevertheless, the part of refs. 19 and 24 dealing with the convergence of the beta function series for small enough M [see (14.1)] still applies to our spinning case. Hence also in the spinning case one may hope to learn more from the analysis of the higher order contributions to the beta function. One could look for anomalous behavior: the question, however, deserves a separate analysis and we hope to come back to the problem in a future publication.

Note that the problem looks quite hard because of the many marginal directions: as a dynamical system, this is a rather pathological one due to the resonances associated with the number of marginal directions.

It is unclear if the model introduced by Mattis[30–32] as a spinning variation of the Mattis–Lieb model[26,27] used above can play the same role as the Luttinger model did in the understanding of the spin-0 case. In fact in this model:

$$V = \sum_{\sigma,\sigma' = \pm 1} \int d\mathbf{x}\, d\mathbf{y}\, dt\, d\omega\, d\omega'\, \lambda_0(\mathbf{x} - \mathbf{y}) :\psi^+_{\mathbf{x}t\sigma\omega} \psi^-_{\mathbf{x}t\sigma\omega} : :\psi^+_{\mathbf{y}t\sigma'\omega'} \psi^-_{\mathbf{y}t\sigma'\omega'} : \tag{15.56}$$

so that the relevant part has the form

$$V_L = \sum_{\sigma,\sigma' = \pm 1} \int dx\, d\omega\, d\omega'\, \hat{\lambda}_0(0) :\psi^+_{x,\sigma,\omega} \psi^+_{x,\sigma',\omega'} \psi^-_{x,\sigma',\omega'} \psi^-_{x,\sigma,\omega} : \tag{15.57}$$

This implies

$$y_2 = y_3 = y_4 = 0, \qquad y_1 = y_5 = \hat{\lambda}_0(0) \tag{15.58}$$

These conditions are preserved exactly by the beta function to all orders, so that there is really only one parameter, as in the spin-zero case. For the potential (15.52), on the contrary, one has

$$y_1 = y_5 = \lambda(0), \qquad y_2 = [\lambda(0) + \lambda(2)]/2, \qquad y_3 = y_4 = \lambda(2) \quad (15.59)$$

For this initial condition the dynamical system (15.52) is diverging, unless $\lambda(0) = \lambda(2) = 0$, which in any case is not preserved, if one takes into account also the terms vanishing in the limit $h \to -\infty$.

In conclusion, the ideas of Tomonaga, which are the basis for the Luttinger model, seem to be not easily generalizable to spinning models (see also ref. 29, concluding remarks).

APPENDIX A. PROOF OF (4.2)

Let $p \equiv p_F$, $\beta \equiv p/m$; let p_0 be fixed and $\alpha = \alpha' p_0^{-2} 2^{-2n}$, $\tau = 2^n p_0 t$. Then, denoting by $\Omega_d = 2\pi^{d/2}\Gamma(d/2)^{-1}$ the surface of the d-dimensional unit sphere and denoting by ϑ the angle between $\mathbf{x}$ and $\mathbf{k}$, one has

$$\begin{aligned}
&\tilde{g}_n(\mathbf{x}, t) \\
&= p_0^{-2} 2^{-2n} \int_{1/4}^{1} d\alpha' \int \frac{dk_0\, d^d\mathbf{k}}{(2\pi)^{d+1}} \\
&\quad \times \exp\left\{ -p_0^{-2} 2^{-2n} \alpha' \left[k_0^2 + \frac{(\mathbf{k}^2 - p^2)^2}{4m^2} \right] - ik_0 t - ikx \cos\vartheta \right\} \\
&\quad \times \left(+ik_0 + \frac{\mathbf{k}^2 - p^2}{2m} \right) \\
&= \frac{\Omega_d p_0^{-2} 2^{-2n}}{(2\pi)^{d+1}} \int_{1/4}^{1} d\alpha \int dk_0 \int_{-p}^{+\infty} dh \int d\omega \\
&\quad \times \exp\left\{ -p_0^{-2} 2^{-2n} \alpha \left[k_0^2 + \frac{h^2(h+2p)^2}{4m^2} \right] \right\} \\
&\quad \times \left\{ \exp[-ik_0 t - i(h+p)x \cos\vartheta] \right\} (h+p)^{d-1} \left(+ik_0 + \frac{h(h+2p)}{2m} \right) \\
&= \frac{\Omega_d p_0^d 2^n}{(2\pi)^{d+1}} \int_{1/4}^{1} d\alpha \int dk_0 \int_{-2^{-n}p/p_0}^{\infty} dh \left(\frac{p}{p_0} + 2^n h \right)^{d-1} \\
&\quad \times \exp\left\{ -\left[k_0^2 + h^2 \left(1 + h 2^{n-1} \frac{p_0}{p} \right)^2 \beta^2 \right] \alpha \right\} \\
&\quad \times \left[+ik_0 + h\left(1 + h 2^{n-1} \frac{p_0}{p} \right) \beta \right] [\exp(-ik_0 \tau)]\, J^{(d)}\left[px \left(1 + \frac{p_0}{p} 2^n h \right) \right]
\end{aligned}$$

$$= \frac{\Omega_d \pi^{1/2} p^d 2^n}{(2\pi)^{d+1}} \frac{p_0}{p} \int_{1/4}^{1} \frac{d\alpha}{\alpha^{1/2}} \int_{-2^{-n}p/p_0}^{+\infty} dh \left(1 + 2^n h \frac{p_0}{p}\right)^{d-1}$$
$$\times \left[-\partial_\tau + h\left(1 + 2^{n-1} h \frac{p_0}{p}\right)\beta \right]$$
$$\times \left\{ \exp\left[-\frac{\tau^2}{4\alpha} - h^2 \left(1 + h 2^{n-1} \frac{p_0}{p}\right)^2 \alpha\beta^2 \right] \right\} J^{(d)}\left[px\left(1 + h 2^n \frac{p_0}{p}\right)\right]$$
$$= \int_{1/4}^{1} \frac{d\alpha}{\sqrt{\alpha}} \left(\exp \frac{-\tau^2}{4\alpha}\right) \left[\frac{\tau}{2\alpha} g^0_{\alpha,n}(x) + g^1_{\alpha,n}(x) \right] \tag{A1}$$

where, if J_ν denotes a Bessel function and Γ is the gamma function, we set [see ref. 21, (3.915)], for dimension $d > 1$,

$$J^{(d)}(y) = \int e^{-iy\cos\vartheta}\, d\boldsymbol{\omega}$$
$$= \frac{\Omega_{d-1}}{\Omega_d} \int_0^\pi e^{-iy\cos\vartheta} (\sin\vartheta)^{d-2}\, d\vartheta$$
$$= \frac{\Omega_{d-1}\pi^{1/2}}{\Omega_d} \left(-\frac{2}{y}\right)^{(d-2)/2} \Gamma\left(d - \frac{1}{2}\right) J_{(d-2)/2}(-y) \tag{A2}$$

Note that $J^{(d)}$ is expressed in terms of Bessel functions trivially related to trigonometric functions in the case of d odd. The above equalities hold for $d \geqslant 2$: the case $d = 1$ is easily treated because the integration over ϑ becomes an average over the two values $\vartheta = 0, \pi$.

If $j = 0, 1$,

$$g^j_{\alpha,n}(x) = \frac{\Omega_d \pi^{1/2} p^d 2^n}{(2\pi)^{d+1}} \frac{p_0}{p} \int_{-2^{-n}p/p_0}^{+\infty} dh \left(1 + 2^n h \frac{p_0}{p}\right)^{d-1}$$
$$\times \left[h\left(1 + 2^{n-1} h \frac{p_0}{p}\right)\beta \right]^j$$
$$\times e^{-h^2(1 + h2^{n-1}p_0/p)^2 \alpha\beta^2} J^{(d)}\left[px\left(1 + h 2^n \frac{p_0}{p}\right)\right] \tag{A3}$$

Note that the integrand has a symmetry in h around $-2^{-n}p/p_0$: this permits us to rewrite (A3) as an integral from $-\infty$ to $+\infty$ provided the constant in front of it is divided by 2. Once the integral is written as an integral over the whole line, we can remark that if $\chi(k) = \pi^{-1/2} \int_{-\infty}^{k} dq \exp -q^2$, so that $\chi(k) + \chi(-k) = 1$, then we can freely insert in the integral the function $2\chi(h + 2^{-n}p/p_0)$. After performing the above transformations we can

develop $J^{(d)}$ via the trigonometric addition formulas and extract from the integral trigonometric functions of px.

Consider the case of odd $d=1, 3$, for simplicity, obtaining

$$\begin{aligned} \bar{g}_n &= 2^n \left[\frac{\sin p\,|\mathbf{x}|}{p\,|\mathbf{x}|}\, g_n^s(\xi, \tau) + 2^n \frac{p_0}{p} \cos p\,|\mathbf{x}|\; g_n^c(\xi, \tau) \right], && d=3 \\ \bar{g}_n &= 2^n [\cos px\; g_n^s(\xi, \tau) - \sin px\; g_n^c(\xi, \tau)], && d=1 \end{aligned} \tag{A4}$$

where $\xi = 2^n p_0 \mathbf{x}$ and we have defined

$$\begin{aligned} g_n^\sigma(\xi) = {} & \frac{\Omega_d \pi^{1/2} p^d p_0}{(2\pi)^{d+1}\, p} \\ & \times \int_{1/4}^{1} \frac{d\alpha}{\alpha^{1/2}} \int_{-\infty}^{+\infty} dh \left(1 + 2^n h \frac{p_0}{p}\right)^{d-1} \\ & \times \left[\frac{\tau}{2\alpha} + h\left(1 + 2^{n-1} h \frac{p_0}{p}\right)\beta\right] \chi\left(h + 2^{-n} \frac{p_0}{p}\right) \\ & \times e^{-\tau^2/4\alpha} e^{-h^2(1 + h 2^{n-1} p_0/p)^2\, \alpha\beta^2} \\ & \times \begin{cases} \begin{cases} \sin(h\,|\xi|)/[\,|\xi|\,(1 + 2^n h p_0/p)] & \text{if } \sigma = c \text{ and } d=3 \\ \cos(h\,|\xi|)/(1 + 2^n h p_0/p) & \text{if } \sigma = s \text{ and } d=3 \end{cases} \\ \begin{cases} \cos h\xi & \text{if } \sigma = s \text{ and } d=1 \\ \sin h\xi & \text{if } \sigma = c \text{ and } d=1 \end{cases} \end{cases} \end{aligned} \tag{A5}$$

We get the bounds in Section 4 simply by breaking the trigonometric expressions inside the integrals via the Euler relations and, finally, shifting upward or downward the h integral to a line with constant nonzero imaginary part.

The function $g_n(\xi, \tau, \boldsymbol{\omega})$ defined in (4.5) becomes, with the help of (4.8),

$$\begin{aligned} g_n(\xi, \tau, \boldsymbol{\omega}) &= g_n^s(\xi, \tau) + 2^n (p_0/p)\; g_n^c(\xi, \tau) - i\boldsymbol{\omega}\xi g_n^c(\xi, \tau), && d=3 \\ g_n(\xi, \tau, \omega) &= g_n^s(\xi, \tau) - i\omega g_n^c(\xi, \tau) && d=1 \end{aligned} \tag{A6}$$

where $\boldsymbol{\omega} \in S^2$ if $d=3$ and $\omega \in \{-1, +1\} \equiv S^0$ if $d=1$.

If $n \leqslant 0$, the functions $g_n(\xi_n, \tau_n)$, with $\xi_n = 2^n \mathbf{x} p_0$, $\tau_n = 2^n t p_0$, can be written, for any pre-fixed $r \geqslant 0$,

$$g_n = g_n^0 + 2^n g_n^1 + 2^{2n} g_n^2 + \cdots + 2^{rn} g_n^r \tag{A7}$$

with g^i having the form of a polynomial times a superposition of Gaussians for $i=1,\dots,r-1$. Furthermore, if $z_n \equiv \tau_n^2 + \xi_n^2/\beta^2$, the g_n^i satisfy the bounds

$$|g_n^i| \leqslant C_{r,\kappa} \exp[-\kappa(z_n)^{1/2}], \qquad i=1,\dots,r \tag{A8}$$

The leading term in (A7) is

$$g_n^0 = \frac{\Omega_d \pi p_F^d}{2(2\pi)^{d+1}} \frac{p_0}{\beta p_F} (\tau_n - i\beta^{-1}\boldsymbol{\omega}\xi_n)\, \gamma_0(z_n) \tag{A9}$$

which comes from an evaluation of the integrals in (A5) when $n \to -\infty$, with

$$\gamma_j(z) = \int_{1/4}^{1} e^{-z/4\alpha} \frac{d\alpha}{\alpha^{2+j}} \tag{A10}$$

The analysis of the case $n>0$ is algebraically similar. Since we are not interested in decomposing $\sum_1^\infty \bar{g}_n$ into its components, we can use for it the expressions (A3), (A5) with $n=0$ and the α integral extended between 0 and 1/4, rather than between 1/4 and 1.

APPENDIX B. FOURFOLD INTEGRALS OVER THE FERMI SPHERE

Let $\boldsymbol{\omega}_1$, $\boldsymbol{\omega}_3$, $\boldsymbol{\omega}$, $\boldsymbol{\omega}'$ be unit vectors and $\boldsymbol{\Omega} = \boldsymbol{\omega}_1 + \boldsymbol{\omega} - \boldsymbol{\omega}_3 - \boldsymbol{\omega}'$, $\boldsymbol{\Delta} = \boldsymbol{\omega}_1 - \boldsymbol{\omega}_3$. We introduce a test function F and consider the integrals in (13.22).

To find the appropriate coordinates, we write

$$\boldsymbol{\omega}_0 = \frac{\boldsymbol{\omega}+\boldsymbol{\Delta}}{|\boldsymbol{\omega}+\boldsymbol{\Delta}|}, \qquad \boldsymbol{\omega}' = \boldsymbol{\omega}_0 + \boldsymbol{\rho}, \qquad |\boldsymbol{\omega}+\boldsymbol{\Delta}| = 1+\sigma \tag{B1}$$

and denote by ϑ_1, φ_1 the polar coordinates of $\boldsymbol{\rho}$ in a frame with z axis parallel to $\boldsymbol{\omega}_0$. Then

$$\boldsymbol{\rho}\cdot\boldsymbol{\omega}_0 = \rho\cos\vartheta_1, \qquad \rho = -2\cos\vartheta_1, \qquad \boldsymbol{\rho}\cdot\boldsymbol{\omega}_0 = -\rho^2/2 \tag{B2}$$

The volume element for $\boldsymbol{\omega}'$ is then

$$d\boldsymbol{\omega}' = 2\rho \sin\vartheta_1 \frac{d\vartheta_1\, d\varphi_1}{4\pi} = \frac{\rho\, d\rho\, d\varphi_1}{4\pi} = \frac{d^2\boldsymbol{\rho}_1}{4\pi} \tag{B3}$$

where $\boldsymbol{\rho}_1$ is a vector with modulus $|\boldsymbol{\rho}_1| = \rho$ and anomaly φ_1, to be thought of as lying in the plane tangent to the sphere in $\boldsymbol{\omega}_0$.

The vector $\boldsymbol{\omega}$ will be described in a system of coordinates on the sphere with z axis parallel to $\boldsymbol{\Delta}$. Let (ϑ_2, φ_2) be such coordinates; using

$$\begin{aligned}(1+\sigma)^2 &= 1+\Delta^2+2\boldsymbol{\omega}\cdot\boldsymbol{\Delta}\\ &= 1+\Delta^2+2\Delta\cos\vartheta_2\\ (1+\sigma)\,d\sigma &= \Delta d(\cos\vartheta_2)\end{aligned} \tag{B4}$$

we find

$$d\boldsymbol{\omega} = \sin\vartheta_2\,\frac{d\vartheta_2\,d\varphi_2}{4\pi} \equiv \frac{1+\sigma}{4\pi\Delta}\,d\sigma\,d\varphi_2 \tag{B5}$$

The integration domains and the volume elements are

$$0 \leqslant |\boldsymbol{\rho}_1| \leqslant 2, \qquad |\sigma| \leqslant \Delta, \qquad d\boldsymbol{\omega}\,d\boldsymbol{\omega}' \equiv \frac{1+\sigma}{\Delta}\,\frac{d\varphi_2\,d\sigma\,d^2\boldsymbol{\rho}_1}{(4\pi)^2} \tag{B6}$$

The main variables are

$$\begin{aligned}\boldsymbol{\Omega} &= \boldsymbol{\omega}+\boldsymbol{\Delta}-\boldsymbol{\omega}' = \sigma\boldsymbol{\omega}_0-\boldsymbol{\rho}\\ \boldsymbol{\Omega}^2 &= \sigma^2+\rho^2-2\sigma\boldsymbol{\rho}\cdot\boldsymbol{\omega}_0 = \sigma^2+(1+\sigma)\,\rho_1^2\\ \boldsymbol{\omega}'\cdot\boldsymbol{\Omega} &= \left(\sigma-\frac{1+\sigma}{2}\,\rho_1^2\right), \qquad \boldsymbol{\omega}\cdot\boldsymbol{\Omega} = (\boldsymbol{\omega}\cdot\boldsymbol{\omega}_0\sigma-\boldsymbol{\rho}\cdot\boldsymbol{\omega})\end{aligned} \tag{B7}$$

and the integral becomes

$$\begin{aligned}&\int_{|\sigma|\leqslant\Delta}(1+\sigma)\,\frac{d\sigma}{\Delta}\int_{|\boldsymbol{\rho}_1|\leqslant 2}d\boldsymbol{\rho}_1\int\frac{d\varphi_2}{(4\pi)^2}\,[N^2e^{-N^2[\sigma^2+(1+\sigma)\rho_1^2]}]\\ &\quad\times\left[\frac{(\boldsymbol{\omega}-\boldsymbol{\omega}')^2}{4}+N^2\boldsymbol{\omega}\cdot\boldsymbol{\Omega}\boldsymbol{\omega}'\cdot\boldsymbol{\Omega}\right]F\end{aligned} \tag{B8}$$

The leading behavior for $N\to\infty$ is obtained, therefore, by setting $\sigma=0$, $\boldsymbol{\rho}_1=\mathbf{0}$ in the regular parts of (B8). In this case the vectors $\boldsymbol{\omega}$, $\boldsymbol{\omega}'$ are forced into a *flag* configuration:

$$\begin{aligned}&\boldsymbol{\omega}' = R_{\varphi_2}\boldsymbol{\omega}_1, \qquad \boldsymbol{\omega} = R_{\varphi_2}\boldsymbol{\omega}_3, \qquad \boldsymbol{\omega}-\boldsymbol{\omega}' = -\boldsymbol{\Delta}\\ &\boldsymbol{\omega}\cdot\boldsymbol{\omega}_0 \equiv \boldsymbol{\omega}_1\cdot\boldsymbol{\omega}_3, \qquad \boldsymbol{\omega}'\cdot\boldsymbol{\Omega}\boldsymbol{\omega}\cdot\boldsymbol{\Omega} = \boldsymbol{\omega}_1\cdot\boldsymbol{\omega}_3\sigma^2+\cdots\\ &[\tfrac{1}{4}(\boldsymbol{\omega}-\boldsymbol{\omega}')^2+N^2\boldsymbol{\omega}\cdot\boldsymbol{\Omega}\boldsymbol{\omega}'\cdot\boldsymbol{\Omega}] = [\tfrac{1}{4}(\boldsymbol{\omega}_1-\boldsymbol{\omega}_3)^2+\boldsymbol{\omega}_1\cdot\boldsymbol{\omega}_3N^2\sigma^2+\cdots]\end{aligned} \tag{B9}$$

and the integral becomes, to leading order, (13.24).

The corrections to (B9) have relative order $O(N^{-1})$. The above argument also leads to a rigorous bound on the integral of the form $[J(N|\omega_1-\omega_3|)+O(N^{-1})]\max|F|$ with

$$J(x)=\frac{1}{x}\int_0^x y^2 e^{-y^2}\leqslant\frac{x^2}{1+x^3} \tag{B10}$$

Using the spherical harmonics expansion of λ in a system of polar coordinates with Δ parallel to the z axis, in which $\omega_1=(\vartheta,0)$, $\omega_3=(\pi-\vartheta,0)$, we can write (13.23)

$$\frac{1}{4}\sum_{ll'mm'}\lambda_l\lambda_{l'}Y_{l,m}(\vartheta,0)^2\,Y_{l',m'}(\vartheta,0)^2\,[e^{-im\varphi}+(-1)^m](e^{im'\varphi}+1)(-1)^{m'} \tag{B11}$$

so that

$$\begin{aligned}\langle F\rangle=&\frac{1}{4}\sum_{ll'mm'}\lambda_l\lambda_{l'}Y_{l,m}(\vartheta,0)^2\,Y_{l',m'}(\vartheta,0)^2\\&\times[\delta_{mm'}(-1)^m+\delta_{m0}(-1)^{m'}+\delta_{m'0}(-1)^m+(-1)^{m+m'}]\end{aligned} \tag{B12}$$

which can be used for an approximate analysis of the flow equations to second order.

The *three-quasimomentum inequality* of Section 11 is a simple consequence of the change of variables leading to (B6). In fact, using (B6) with $N=2^{-h}$, we find, for suitable constants C_j and for any Δ,

$$\begin{aligned}&\int|\delta_h(\omega-\omega'+\Delta)|\,d\omega\,d\omega'\\&\quad\leqslant C_1\int_0^{\Delta}\frac{1+\sigma}{\Delta}\,d\sigma\,d^2\rho_1\,N^2e^{-\kappa N^2[\sigma^2+(1+\sigma)\rho_1^2]}\\&\quad\leqslant\frac{C_2}{1+N\Delta}\end{aligned} \tag{B13}$$

where we use, for simplicity, the bound on the leading term in the propagator g_h generating δ_h (which explains the Gaussian term), and we denote $|\Delta|$ as Δ.

Hence, if $\Delta=\omega_2-\mathbf{k}$, $\mathbf{k}=|\mathbf{k}|\,\omega_0$, $\omega_2=\omega_0+\rho$, using (B3), we find

$$\begin{aligned}&\int|\delta_h(\omega-\omega'+\omega_2-\mathbf{k})|\,d\omega\,d\omega'\,d\omega_2\\&\quad\leqslant C_3\int\frac{d^2\rho}{1+N|\rho|}\\&\quad\leqslant\frac{C_4}{N}\int_0^2 d\rho\,\frac{N\rho}{1+N\rho}\leqslant\frac{2C_4}{N}\end{aligned} \tag{B14}$$

proving (11.9).

Another important inequality used in the derivation of the bound (11.26) is

$$\left| 2^h \int d\omega\, dx\, (2^h t - i 2^h \beta^{-1} \omega \cdot \mathbf{x})\, e^{ip_F(\omega - \omega_0)\mathbf{x}} \gamma_0(2^{2h} x^2) \right| \leqslant C \qquad \text{(B15)}$$

where $\gamma_0(z)$ is defined by (A8) and $x^2 = t^2 + \mathbf{x}^2/\beta^2$. In the lhs of (B15) the integral of the term proportional to t is exactly zero and the remaining part can be bounded by

$$C_1 2^{-4h} \int_{1/4}^{1} \frac{d\alpha}{\alpha^2} \int d\omega\, (1 - \omega \cdot \omega_0)\, e^{-2\alpha\beta^2 2^{-2h}(1 - \omega \cdot \omega_0)} \leqslant C \qquad \text{(B16)}$$

APPENDIX C. A HEURISTIC ANALYSIS OF THE FLOW OF THE RUNNING COUPLINGS

In this section we make precise the source of our interpretation that the running couplings flow indicates that the pair interaction evolves diverging as a deltalike interaction with integral $2^{(d-1)h}$, i.e., as 2^{-h} at its maximum.

We deduce from (A7) that, to leading order in n, the propagator $g^{(\leqslant n)}(x, \omega, p_F, \beta)$ (we add the explicit dependence of the propagator on p_F and β, as the coming analysis is based on rescalings) can be written

$$\begin{aligned} & g^{(\leqslant n)}(x, \omega, p_F, \beta) \\ & \quad = \sum_{-\infty}^{n} g_h(\xi_h) \\ & \quad \simeq \frac{\Omega_d \pi p_F^d}{2(2\pi)^{d+1}} \frac{p_0}{p_F} \beta 2^n \sum_{h=-\infty}^{0} 2^h (\tau_{n+h} - i\beta^{-1} \omega \xi_{n+h})\, \gamma_0(z_{n+h}) \\ & \quad \simeq \frac{\Omega_d \pi p_F^{d-1}}{2(2\pi)^{d+1}} \beta \frac{t - i\beta^{-1} \omega \mathbf{x}}{t^2 + \mathbf{x}^2/\beta^2} G(2^n |x|\, p_0) \end{aligned} \qquad \text{(C1)}$$

where $|x|^2 = t^2 + \mathbf{x}^2/\beta^2$, and

$$G(y) = \sum_{h=-\infty}^{0} (2^h y)^2\, \gamma_0((2^h y)^2) \qquad \text{(C2)}$$

Hence

$$g^{(\leqslant n)}(x, \omega, p_F, \beta) = 2^{nd} g^{(\leqslant 0)}(2^n \mathbf{x}, \omega, 2^{-n} p_F, \beta) \qquad \text{(C3)}$$

so that

$$\psi^{(\leqslant n)}_{x, \omega, p_F, \beta} = 2^{nd/2} \psi^{(\leqslant 0)}_{2^n x, \omega, 2^{-n} p_F, \beta} \qquad \text{(C4)}$$

Equation (C4) permits us to find, heuristically and for the purpose of

motivating the intuitive statements of the Introduction and of Section 11, the relation between the running coupling constants in the quasiparticle formalism and the pair potential between the particles.

We argue as follows: in a *first*-order calculation the relevant part of the interaction remains the same:

$$V^{(n)} = \int \lambda_0(\omega_1, ..., \omega_4)\, e^{i(\omega_1+\omega_2-\omega_3-\omega_4)\,x p_F}\, d\omega_1 \cdots dx$$
$$\times :\psi^{+(\leqslant n)}_{x,\omega_1,p_F,\beta} \psi^{+(\leqslant n)}_{x,\omega_2,p_F,\beta} \psi^{-(\leqslant n)}_{x,\omega_3,p_F,\beta} \psi^{-(\leqslant n)}_{x,\omega_4,p_F,\beta}: \tag{C5}$$

On the other hand, (C4) tells us that

$$V^{(n)} = \int \lambda_0(\omega_1,...)\, 2^{2nd} :\psi^{+(\leqslant 0)}_{2^n x,\omega_1,2^{-n}p_F,\beta} \cdots :$$
$$\times\, e^{i(\omega_1+\omega_2-\omega_3-\omega_4)\,x p_F}\, dx\, d\omega_1 \cdots$$
$$= \int \lambda_0(\omega_1,...)\, 2^{n(d-1)} :\psi^{+(\leqslant 0)}_{x,\omega_1,2^{-n}p_F,\beta} \cdots :$$
$$\times\, e^{i(\omega_1+\omega_2-\omega_3-\omega_4)\,x 2^{-n} p_F}\, dx\, d\omega_1 \cdots \tag{C6}$$

Hence, we read (C6) by saying that the pair potential on scale n is, to first order and measuring the length scales with a unit 2^{-n} bigger than the initial one, a pair potential $w_n(\mathbf{x})$ between fermions in a state with a Fermi surface at $2^{-n}p_F$, mass $m2^{-n}$ (so that β is unchanged), and such that [see (6.2), (6.4)]

$$w_n(\mathbf{x}) = \lambda_0(2^{-n}\mathbf{x})\, 2^{-n} \tag{C7}$$

Therefore our potential is a δ-like potential and is precisely an approximate delta function with width $2^n r_0$, r_0 being the range of λ_0, and integral proportional to $2^{(d-1)n}$; we write it as

$$w_n(\mathbf{x}) = 2^{(d-1)n} V_0 \delta_n(\mathbf{x}) \tag{C8}$$

where V_0 is the integral of λ_0, so the integral of the deltalike potential is $V_0 2^{(d-1)n}$, very small if $n \to -\infty$ and $d > 1$, but not small enough to keep the potential bounded, $\forall d \geqslant 1$.

This divergence of the maximum of the potential forces us to abandon the formalism of the initial particle fields and to adopt, even in $d = 1$, the quasiparticle formalism.

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